Calculus 1
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Derivatives of Exponential, Logarithmic and Trigonometric Functions

Derivative of the inverse function. If $f(x)$ is a one-to-one function (i.e. the graph of $f(x)$ passes the horizontal line test), then $f(x)$ has the inverse function $f^{-1}(x)$. Recall that f and f^{-1} are related by the following formulas

$$y = f^{-1}(x) \Longleftrightarrow x = f(y).$$

Also, recall that the graphs of $f^{-1}(x)$ and $f(x)$ are symmetrical with respect to line $y = x$.

Some pairs of inverse functions you encountered before are given in the table below where n is a positive integer and a is a positive real number.

f	x^2	x^n	e^x	a^x
$\mathbf{f^{-1}}$	$\sqrt{x}$	$\sqrt[n]{x}$	$\ln x$	$\log_a x$

With $y = f^{-1}(x)$, $\frac{dy}{dx}$ denotes the derivative of f^{-1} and since $x = f(y)$, $\frac{dx}{dy}$ denotes the derivative of f. Since the reciprocal of $\frac{dy}{dx}$ is $\frac{dx}{dy}$ we have that

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}.$$

Thus, the derivative of the inverse function of f is reciprocal of the derivative of f.

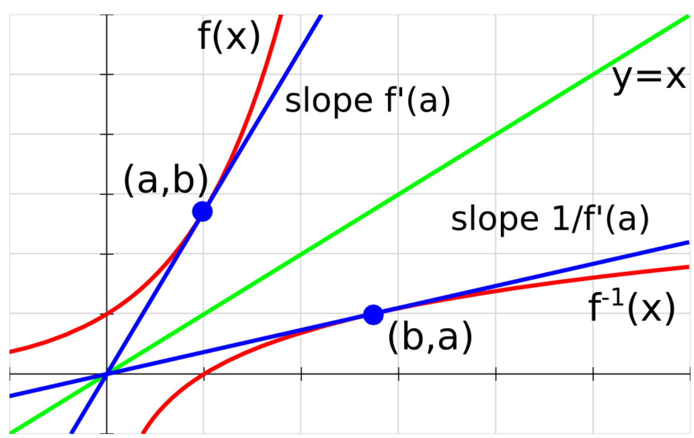
Another way to see this is to consider relation

$$f(f^{-1}(x)) = x \text{ or } f^{-1}(f(x)) = x,$$

and to differentiate any of these identities. For example, differentiating $f^{-1}(f(x)) = x$ and using the chain rule for the left hand side produces

$$(f^{-1})'(f(x)) \cdot f'(x) = 1 \Longrightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Graphically, this rule means that



> The slope of the tangent to $f^{-1}(x)$ at point (b, a) is reciprocal to the slope of the tangent to $f(x)$ at point (a, b).

Example 1. If $f(x)$ has the inverse, $f(2) = 1$, and $f'(2) = 3$, find $(f^{-1})'(1)$.

Solution. Since f passes the point $(2,1)$, f^{-1} passes the point (1,2). The tangent to $f(x)$ at $x=2$ is 3 so the tangent to f^{-1} at $x=1$ is the reciprocal $\frac{1}{3}$. Hence $(f^{-1})'(1)=\frac{1}{3}$.

Exponential Functions and their derivatives.

In a pre-calculus course you have encountered exponential function a^x of any base $a>0$ and their inverse functions . All these functions can be considered to be a composite of e^u and $x\ln a$ since

$$a^x = e^{\ln a^x} = e^{x\ln a}$$

Thus, using the chain rule and formula for derivative of e^x, you can obtain the formula for derivative of any a^x.

The derivative of e^x can be computed by

$$\frac{de^x}{dx} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h}$$

Before you see a proof of $\lim_{h\to 0}\frac{e^h-1}{h}=1$ in higher calculus courses, you can convince yourself that the limit $\lim_{h\to 0}\frac{e^h-1}{h}$ is 1 by evaluating the quotient $\frac{e^h-1}{h}$ at several values of h close to 0 as in the table below.

$\mathbf{h}$	0.1	0.01	0.001	0.0001
$\mathbf{\frac{e^h-1}{h}}$	1.0517	1.0050	1.0005	1.00005

This indicates that $\lim_{h\to 0}\frac{e^h-1}{h}=1$ and so $\frac{de^x}{dx}=e^x\lim_{h\to 0}\frac{e^h-1}{h}=e^x$. Thus,

> the derivative of e^x is e^x.

Example 2. Find the derivative of the following functions

(a) $y=e^{3x}$ (b) $y=x^2e^{3x}$

Solution. (a) We can consider the function $y=e^{3x}$ as a composite of the outer function e^u and the inner function $3x$.

$y=e^{3x}$	$\Rightarrow$	$y' =$	e^{3x}	$\cdot$	3
		derivative of the composite	derivative of the outer, keep the inner unchanged		derivative of the inner

Thus the derivative is $y'=3e^{3x}$.

(b) The function is a product of $f(x)=x^2$ and $g(x)=e^{3x}$. By part (a), $g'(x)=3e^{3x}$. Since $f'(x)=2x$, the product rule produces $y'=f'g+g'f=2xe^{3x}+x^2(3)e^{3x}=(2x+3x^2)e^{3x}$.

Using the formula $\frac{de^x}{dx}=e^x$ we can obtain the derivative of a^x. Recall that $y=a^x=e^{\ln a^x}=e^{x\ln a}$. In the last step we used the rule $\log_a(x^r)=r\log_a x$. Thus the function $y=a^x=e^{x\ln a}$ can be consider as a composite of e^u and $u=x\ln a$. Since $\ln a$ is a constant $u'=\ln a$ (just as in part (a) of the previous example we had $u=3x\Rightarrow u'=3$). Thus, the derivative of $y=a^x$ is $y'=e^{x\ln a}\ln a$. Note that $e^{x\ln a}$ is the original function $y=a^x$. Thus, $y'=a^x\ln a$ and we have that the derivative of a^x is $a^x \ln a$.

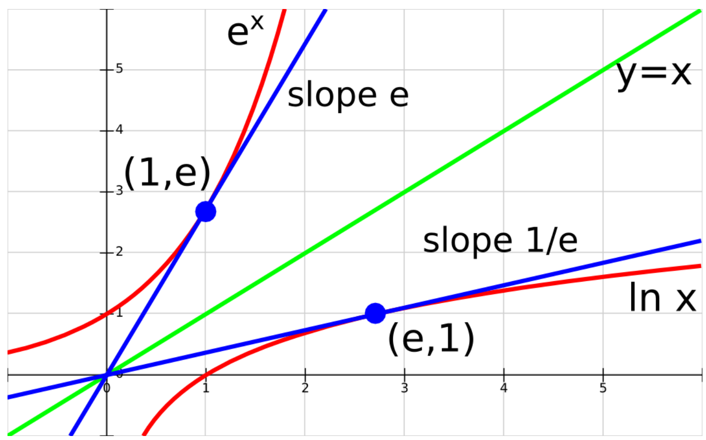
Example 3. Find the derivative of the following functions

(a) $y = 2^{5x+7}$ (b) $y = \dfrac{3^x - 3^{-x}}{2}$

Solution. (a) Consider the function as a composite of 2^u and $u = 5x + 7$. Using the formula for a^x with $a = 2$ obtain $2^u \ln 2 = 2^{5x+7} \ln 2$ for the derivative of the outer. Since the derivative of the inner is 5, $y' = 5 \ln 2\, 2^{5x+7}$.

(b) Note that the function can be written as $y = \frac{1}{2}(3^x - 3^{-x})$. The derivative of the first term in the parenthesis is $3^x \ln 3$ by the formula for derivative of a^x with $a = 3$. Using the chain rule with inner function $-x$, the derivative of the second part 3^{-x} is $3^{-x} \ln 3(-1) = -3^{-x} \ln 3$. Thus $y' = \frac{1}{2}(3^x \ln 3 + 3^{-x} \ln 3) = \frac{\ln 3}{2}(3^x + 3^{-x})$.

Logarithmic function and their derivatives.



Recall that the function $\log_a x$ is the inverse function of a^x : thus $\log_a x = y \Leftrightarrow a^y = x$.

If $a = e$, the notation $\ln x$ is short for $\log_e x$ and the function $\ln x$ is called the **natural logarithm.**

The derivative of $y = \ln x$ can be obtained from derivative of the inverse function $x = e^y$. Note that the derivative x' of $x = e^y$ is $x' = e^y = x$ and consider the reciprocal:

$$y = \ln x \quad \Rightarrow \quad y' = \frac{1}{x'} = \frac{1}{e^y} = \frac{1}{x}.$$

The derivative of logarithmic function of any base can be obtained converting $\log_a$ to ln as $y = \log_a x = \frac{\ln x}{\ln a} = \ln x \frac{1}{\ln a}$ and using the formula for derivative of $\ln x$. So we have

$$\frac{d}{dx} \log_a x = \frac{1}{x} \frac{1}{\ln a} = \frac{1}{x \ln a}.$$

The derivative of $\ln x$ is $\frac{1}{x}$ and the derivative of $\log_a x$ is $\frac{1}{x \ln a}$.

To summarize,

y	e^x	a^x	$\ln x$	$\log_a x$
y′	e^x	$a^x \ln a$	$\frac{1}{x}$	$\frac{1}{x \ln a}$

Example 4. Find the derivative of the following functions

(a) $y = \ln(x^2 + 2x)$ (b) $y = \log_2(3x + 4)$

(c) $y = x \ln(x^2 + 1)$ (d) $y = \ln(x + 5e^{3x})$

Solution. (a) Using the chain rule with the outer $\ln u$ and the inner x^2+2x, you have $y' = \frac{1}{x^2+2x}(2x+2) = \frac{2x+2}{x^2+2x} = \frac{2(x+1)}{x(x+2)}$.

(b) Using the chain rule with the outer $\log_2 u$ and the inner $3x+4$, you have $y' = \frac{1}{\ln 2(3x+4)}\, 3 = \frac{3}{\ln 2(3x+4)}$.

(c) Use the product rule with $f(x) = x$ and $g(x) = \ln(x^2+1)$ and the chain rule with derivative of g with the outer $\ln u$ and the inner x^2+1. Obtain that $y' = f'g + g'f = 1\ln(x^2+1) + \frac{1}{x^2+1}(2x)(x) = \ln(x^2+1) + \frac{2x^2}{x^2+1}$.

(d) Use the chain rule with the outer $\ln u$ and the inner $u = x + 5e^{3x}$. For derivative of the part e^{3x}, you will need to use the chain rule again to obtain $u' = 1 + 5e^{3x}(3) = 1 + 15e^{3x}$. Thus, $y' = \frac{1}{x+5e^{3x}}\left(1+15e^{3x}\right) = \frac{1+15e^{3x}}{x+5e^{3x}}$.

Trigonometric functions and their derivatives.

The derivative of $\sin x$ can be determined using the trigonometric identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$ and calculating the limits $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ using tables. Thus,

$$\begin{aligned}
\frac{d\sin x}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \sin x \, \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \, \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x\,(0) + \cos x\,(1) \\
&= \cos x
\end{aligned}$$

Example 5. Find derivatives of the following functions.

(a) $y = \sin^3 x$ (b) $y = \sin x^3$ (c) $y = x^3 \sin x$

Solution. (a) In order to see better the inner and outer function in this composite, note that the function can be represented also as $y = (\sin x)^3$. In this representation it is more obvious that the outer function is u^3 and the inner is $\sin x$. Thus the chain rule produces $y' = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x$.

(b) This function is a composite of the outer $\sin u$ and the inner x^3. Thus the chain rule produces $y' = \cos x^3 (3x^2) = 3x^2 \cos x^3$.

(c) Using the product rule with $f(x) = x^2$ and $g(x) = \sin x$ we obtain $y' = f'g + g'f = 3x^2 \sin x + \cos x(x^3) = 3x^2 \sin x + x^3 \cos x$.

The derivative of $\cos x$ can be found to be $-\sin x$ either using the trigonometric identity for cosine of a sum and similar arguments as above or the implicit differentiation. In section on Implicit Differentiation we demonstrate this second method. The derivatives of the remaining trigonometric functions can be obtained by expressing these functions in terms of sine or cosine. In general, you can always express a trigonometric function in terms of sine, cosine or both and then use just the following two formulas.

The derivative of $\sin x$ is $\cos x$ and the derivative of $\cos x$ is $-\sin x$.

Example 6. Find derivatives of $\tan x$ and $\sec x$. Simplify your answers.

Solution. Recall that $\tan x = \frac{\sin x}{\cos x}$ so, using the quotient rule with $f(x) = \sin x$ and $g(x) = \cos x$, obtain $\frac{d\tan x}{dx} = \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^x x}{\cos^2 x} = \frac{1}{\cos^2 x}$ or $\sec^2 x$.

Recall that $\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$ so, using the chain rule, obtain that

$$\frac{d \sec x}{dx} = -1(\cos x)^{-2}(-\sin x) = (\cos x)^{-2} \sin x \text{ or } \sec^2 x \sin x.$$

Sometimes this function is also written as $\sec x \tan x$.

Practice problems.

1. Find the derivative of the given functions.
   - (a) $y = e^{3x}(x^3 + 2x - 5)$
   - (b) $y = 3^{2x^2+5}$
   - (c) $y = x\ 5^{3x}$
   - (d) $y = (2x + e^{x^2})^4$
   - (e) $y = \frac{e^{2x}+e^{-2x}}{x^2}$
   - (f) $y = \ln(5x - e^{5x})$
   - (g) $y = \log_3(x^2 + 5)$
   - (h) $y = \log_2(x^2 + 7x)$
   - (i) $y = \sin(2x^2 + 4)$
   - (j) $y = x^2 \cos x^2$
   - (k) $y = \sin 3x \cos 5x$
   - (l) $y = \log_2 x + 3\sin x - xe^x$
   - (m) $y = \cot x^2$
   - (n) $y = e^{\csc x}$

2. Find an equation of the line tangent to the curve at the indicated point.
   - (a) $f(x) = \frac{e^{2x}-1}{e^{2x}+1}$ at $x = 0$.
   - (b) $f(x) = \ln\sqrt{2x-1}$ at $x = 1$.

3. Assume that $f(x)$ is a function differentiable for every value of x.
   - (a) If $F(x) = e^{3f(x)}$, $f(3) = 0$ and $f'(3) = 2$, determine $F'(3)$.
   - (b) If $F(x) = \ln(f(x) + 1)$, $f(1) = 0$ and $f'(1) = 1$, determine $F'(1)$.
   - (c) If $f(x)$ has the inverse and $f(3) = 2$ and $f'(3) = 6$, find $(f^{-1})'(2)$.

4. A **differential equation** is an equation in unknown function that contains one or more derivatives of the unknown function. For example, $y'^2 + y = \sin x$ and $y'' + \sin(xy) = 0$ are differential equations.
   - (a) Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2y = 6x^2$.
   - (b) Show that $y = \frac{1}{x+c}$ is a solution of the differential equation $y' = -y^2$ for every value of the constant c.
   - (c) Show that $y = ce^{2x}$ is a solution of the differential equation $y'' - 3y' + 2y = 0$ for every value of the constant c.
   - (d) Show that $y = c_1 \cos 2x + c_2 \sin 2x$ is a solution of differential equation $y'' + 4y = 0$ for every value of the constants c_1 and c_2.
   - (e) Find a value of the constant A for which the function $y = Ae^{3x}$ is a solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

5. The concentration of pollutants (in grams per liter) in a river is approximated by $C(x) = .04e^{-4x}$ where x is the number of miles downstream from a place where the measurements are taken.

   (a) Determine the initial pollution and the pollution 2 miles downstream.

   (b) Determine how much the concentration changed on average within the first two miles.

   (c) Determine how fast the concentration changes 2 miles downstream.

6. A mass at the end of a vertical spring is stretched 5 cm beyond its natural length and is released. If s measures the length from the natural length, the position at time t (in seconds) can be described by $s(t) = 5\cos 2t$.

   (a) Find the velocity and acceleration at time t and graph the position, velocity and acceleration on the same plot for one period of the motion.

   (b) Mark the intervals on which the object speeds up and the intervals on which it speeds down on the plot in part (a).

   (c) Determine the times when the mass returns to equilibrium position.

   (d) Determine the position, velocity, and acceleration 4 second after the object is released. Using that data, determine the direction in which the object is moving and if it is speeding up or slowing down. Do the same for $t = 5$ minutes after the motion started.

Solutions.

1. (a) Use the product rule with $f(x) = e^{3x}$ and $g(x) = x^3+2x-5$ and the chain for $f'(x) = e^{3x}(3)$ so that $y' = 3e^{3x}(x^3 + 2x - 5) + (3x^2 + 2)e^{3x}$.

   (b) Use the chain rule with inner $2x^2 + 5$ so that $y' = 3^{2x^2+5} \cdot \ln 3 \cdot 4x = 4x \ln 3\, 3^{2x^2+5}$.

   (c) Use the product rule with $f(x) = x$ and $g(x) = 5^{3x}$ and the chain for $g'(x) = 5^{3x} \ln 3(3)$ so that $y' = 5^{3x} + 3x \ln 5\, 5^{3x}$.

   (d) The chain rule with inner $2x + e^{x^2}$ and another chain rule for derivative of e^{x^2} produces $y' = 4(2x + e^{x^2})^3 \cdot (2 + e^{x^2}2x) = 8(1 + xe^{x^2})(2x + e^{x^2})^3$.

   (e) The quotient rule with $f(x) = e^{2x}+e^{-2x}$ and $g(x) = x^2$ and the chain for $f'(x) = 2e^{2x}-2e^{-2x}$ produces $y' = \frac{(2e^{2x}-2e^{-2x})x^2-2x(e^{2x}+e^{-2x})}{x^4} = \frac{2x((x-1)e^{2x}-(x+1)e^{-2x})}{x^4} = \frac{2((x-1)e^{2x}-(x+1)e^{-2x})}{x^3}$.

   (f) Use the chain rule with inner $5x - e^{5x}$ and another chain rule for derivative of e^{5x} so that $y' = \frac{1}{5x-e^{5x}}(5 - 5e^{5x}) = \frac{5-5e^{5x}}{5x-e^{5x}}$.

   (g) The chain rule with the outer $\log_3 u$ and the inner x^2+5 produces $y' = \frac{1}{\ln 3(x^2+5)\, 2x} = \frac{2x}{\ln 3(x^2+5)}$.

   (h) The chain rule with the outer $\log_2 u$ and the inner $x^2 + 7x$ produces $y' = \frac{1}{\ln 2(x^2+7x)\,(2x+7)} = \frac{2x+7}{\ln 2(x^2+7x)}$.

   (i) The chain rule with the outer $\cos u$ and the inner $2x^2+4$ produces $y' = \cos(2x^2+4)\cdot(4x) = 4x\cos(2x^2 + 4)$.

   (j) The product rule with $f(x) = x^2$ and $g(x) = \cos x^2$ and the chain for $g'(x) = -\sin x^2(2x)$ so that $y' = 2x\cos x^2 - \sin x^2(2x)(x^2) = 2x\cos x^2 - 2x^3 \sin x^2$.

   (k) The product rule with $f(x) = \sin 3x$ and $g(x) = \cos 5x$ and the chain for $f'(x) = \cos 3x(3)$ and $g'(x) = -\sin 5x(5)$ so that $y' = 3\cos 3x \cos 5x - 5\sin 5x \sin 3x$.

(l) Since the function is a sum of three terms, you can differentiate term by term. Use the product rule for the last term. Obtain $y' = \frac{1}{x\ln 2} + 3\cos x - e^x - xe^x$.

(m) Representing the function as $y = \cot x^2 = \frac{\cos x^2}{\sin x^2}$ and using the quotient rule with $f(x) = \cos x^2$ and $g(x) = \sin x^2$ and the chain for $f'(x) = -\sin x^2(2x)$ and $g'(x) = \cos x^2(2x)$ obtain that $y' = \frac{-\sin x^2(2x)\sin x^2 - \cos x^2(2x)\cos x^2}{\sin^2 x^2} = \frac{-2x(\sin^2 x^2+\cos^2 x^2)}{\sin^2 x^2} = \frac{-2x}{\sin^2 x^2}$.

(n) Use the chain rule with the outer e^u and the inner $u = \csc x = \frac{1}{\sin x} = (\sin x)^{-1}$ so that $u' = -1(\sin x)^{-2}\cos x$ and obtain that $y' = e^{\csc x} - 1(\sin x)^{-2}\cos x$ or $y' = -e^{\csc x}\csc^2 x\cos x$.

2. (a) Use the quotient rule to find $f'(x) = \frac{2e^{2x}(e^{2x}+1) - 2e^{2x}(e^{2x}-1)}{(e^{2x}+1)^2}$ and evaluate it at $x = 0$ to get the slope $f'(0) = \frac{2(1+1)-2(1-1)}{(1+1)^2} = \frac{4-0}{4} = 1$ so the slope is 1. Since $f(0) = \frac{1-1}{1+1} = 0$, the tangent is $y - 0 = 1(x - 0) \Rightarrow y = x$.

   (b) Either represent the function as $f(x) = \ln(2x-1)^{1/2} = \frac{1}{2}\ln(2x-1)$ and use a single chain rule to get $f'(x) = \frac{1}{2}\frac{1}{2x-1}(2) = \frac{1}{2x-1}$ or keep it as $f(x) = \ln(2x-1)^{1/2}$ and use two chain rules to obtain $f'(x) = \frac{1}{(2x-1)^{1/2}}\frac{1}{2}(2x-1)^{-1/2}(2) = \frac{1}{(2x-1)^{1/2}}\frac{1}{(2x-1)^{1/2}} = \frac{1}{2x-1}$. In either case $f'(1) = \frac{1}{2-1} = 1$ so the slope is 1. Since $f(1) = \ln\sqrt{2-1} = \ln 1 = 0$, the tangent line is $y - 0 = 1(x-1) \Rightarrow y = x - 1$.

3. (a) Use the chain rule with the outer e^u and the inner $3f(x)$ to find the derivative of $F(x) = e^{3f(x)}$ to be $F'(x) = e^{3f(x)}3f'(x)$. Since $f(3) = 0$ and $f'(3) = 2$, $F'(3) = e^0\,3\,(2) = 6$.

   (b) Use the chain rule with the outer $\ln u$ and the inner $f(x) + 1$ to find the derivative of $F(x) = \ln(f(x)+1)$ to be $F'(x) = \frac{f'(x)}{f(x)+1}$. Since $f(1) = 0$ and $f'(1) = 1$, $F'(1) = \frac{1}{0+1} = 1$.

   (c) Since $f(3) = 2$, $(f^{-1})'(2) = \frac{1}{f'(3)}$. Then since $f'(3) = 6$, $(f^{-1})'(2) = \frac{1}{f'(3)} = \frac{1}{6}$.

4. (a) $y = x^2 \Rightarrow y' = 2x$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow 2x + 3x^2(x^2) = 6x^2 \Rightarrow 2x + 3x^4 = 6x^2$. This equation does not hold for every value of x (for example if $x = 1$ the equation false identity $2 + 3 = 6$) so $y = x^2$ is not a solution of the given equation.

   $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow -3x^2e^{-x^3} + 3x^2(2 + e^{-x^3}) = 6x^2 \Rightarrow -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 \Rightarrow 6x^2 = 6x^2$. This identity holds for every x so the given function is a solution of the equation.

   (b) Find the derivative of $y = \frac{1}{x+c}$ to be $y' = \frac{-1}{(x+c)^2}$ and plug the function and its derivative into the equation $y' = -y^2 \Rightarrow \frac{-1}{(x+c)^2} = -\left(\frac{1}{x+c}\right) \Rightarrow \frac{-1}{(x+c)^2} = \frac{-1}{(x+c)^2}$. This identity holds for every x so the given function is a solution of the equation.

   (c) Find the derivatives of $y = ce^{2x}$ to be $y' = 2ce^{2x}$ and $y'' = 4ce^{2x}$ and plug into the equation $y'' - 3y' + 2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4 - 6 + 2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

   (d) $y = c_1\cos 2x + c_2\sin 2x \Rightarrow y' = -2c_1\sin 2x + 2c_2\cos 2x \Rightarrow y'' = -4c_1\cos 2x - 4c_2\sin 2x$. Plugging into the differential equation $y'' + 4y = 0$ gives you $-4c_1\cos 2x - 4c_2\sin 2x + 4c_1\cos 2x + 4c_2\sin 2x = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

   (e) Find the derivatives of $y = Ae^{3x}$ to be $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$ and substitute them into the equation $y'' - 3y' + 2y = 6e^{3x}$ to get

$$9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3.$$

Thus, $y = 3e^{3x}$ is a solution of differential equation.

5. (a) The initial pollution is $C(0) = 0.04$ and the pollution 2 miles downstream is $C(2) \approx 1.3\cdot 10^{-5}$ grams per liter.

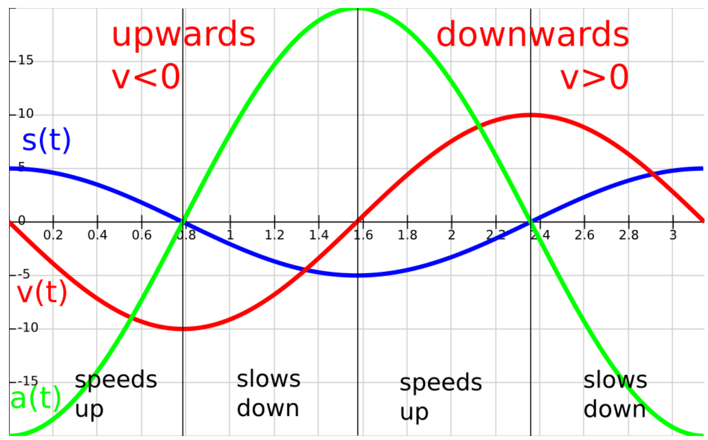
   (b) The average rate of change within first two miles is $\frac{C(2)-C(0)}{2-0} \approx -.01999 \approx -.02$. Thus the concentration is decreasing on average by .02 grams per liter per mile during the first two miles.

   (c) The derivative is $C'(x) = .04e^{-4x}(-4) = -.16e^{-4x}$ so that $C'(2) = -5.37 \cdot 10^{-5}$, thus the concentration is decreasing by .0000537 grams per liter per mile 2 miles downstream.

6. (a) Differentiate $s(t) = 5\cos 2t$ to find the velocity $v(t) = s'(t) = -10\sin 2t$.

   Differentiate the velocity to find the acceleration $a(t) = v'(t) = s''(t) = -20\cos 2t$.These periodic functions have period $\frac{2\pi}{2} = \pi$ so you can graph them on the interval $[0, \pi]$. Since the amplitudes of s, v and a are 5, 10 and 20, respectively, you can choose the range [-20, 20] so that all the functions are displayed.

   (b) The object speeds up on intervals where v and a have the same sign and slows down on intervals where v and a have the opposite sign. These intervals are marked on the graph.



   (c) The the mass returns to equilibrium position when $s = 0$. $5\cos 2t = 0 \Rightarrow \cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots \Rightarrow t = \pi, 3\pi, 5\pi \ldots$

   (d) $s(4) = 5\cos 8 \approx -0.73$ cm meaning that the object is 0.75 above the equilibrium position. The velocity is $v(4) = -10\sin 8 \approx -9.89$ cm per second and, since it is negative, the object is moving upwards. $a(4) = -20\cos 8 \approx 2.91$ cm per second squared. Since the velocity and acceleration have different signs, the object is slowing down.

   When $t = 5$ min $= 300$ sec, $s(300) = 5\cos 600 \approx -4.995 \approx -5$ cm meaning that the object is almost at the highest point above the equilibrium position. The velocity is $v(300) = -10\sin 600 \approx -0.44$ cm per second. The small value of velocity agrees with the fact that the object is almost at the highest point. The negative sign indicates it is still moving upwards. $a(300) = -20\cos 600 \approx 19.98$ cm per second squared. Since the velocity and acceleration have different signs, the object is slowing down.