

## The Fundamental Theorem of Calculus. The Total Change Theorem and the Area Under a Curve.

Example 2 of the previous section we approximated the distance traveled during some time interval and given velocities at certain points of the interval using the left and right sums. Since the left and right sum approximate the definite integral, this suggests that the exact distance traveled can be obtained as the definite integral of the velocity.

More generally, if a continuous function  $f(x)$  represents the rate of change of  $F(x)$  (so that  $F(x)$  is an antiderivative of  $f(x)$ ) on interval  $[a, b]$ , then

the **total change**  $F(b) - F(a)$  in  $F(x)$  as  $x$  changes from  $a$  to  $b$  is given by  $\int_a^b f(x) dx$ .

We illustrate why this is true in case when  $f(x)$  is nonnegative. Consider the area under the graph of  $f$  to be a function of  $x$  in  $[a, b]$  as follows  $A(x) = \int_a^x f(t)dt$  where  $a \leq x \leq b$ . Since  $f$  is nonnegative,  $A(x)$  corresponds to the area between the graph of  $f$  and  $x$ -axis on  $[a, x]$ . Thus  $A(a)$  is zero and  $A(b)$  is the total area  $\int_a^b f(x)dx$ .

We claim that  $A'(x) = f(x)$  so that  $A(x)$  is an antiderivative of  $f(x)$ .

Indeed, if  $h$  is a small positive number, the difference  $A(x+h) - A(x)$  is the area under  $f(x)$  on interval  $[x, x+h]$ . Since  $h$  is small, this area can be approximated by the area of rectangle of height  $f(x)$  with base  $h$ . Thus,

$$A(x+h) - A(x) \approx f(x)h \Rightarrow$$

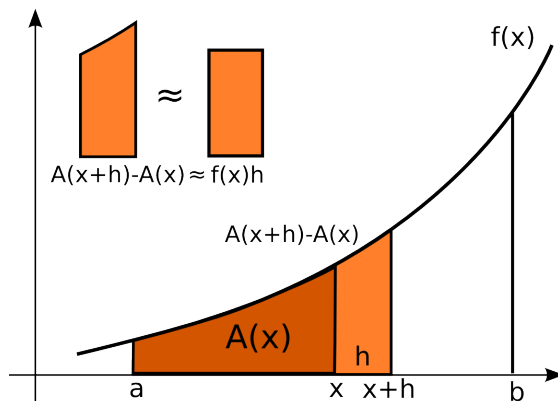
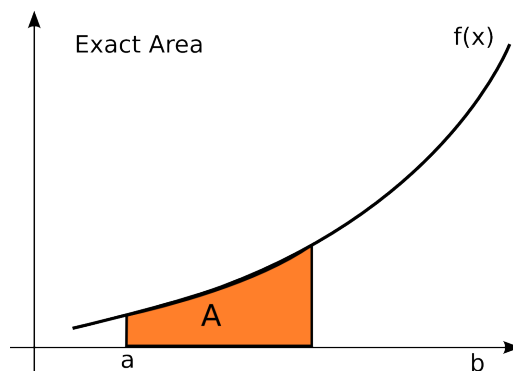
$$\frac{A(x+h) - A(x)}{h} \approx f(x) \Rightarrow$$

$$A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

Thus  $A(x)$  is an antiderivative of  $f(x)$ . If  $F(x)$  is any other antiderivative of  $f(x)$ , then  $F(x) = A(x) + c$  for some constant  $c$ . Since  $A(a) = 0$ , we have that

$$F(b) - F(a) = A(b) + c - A(a) - c = A(b) = \int_a^b f(x)dx.$$

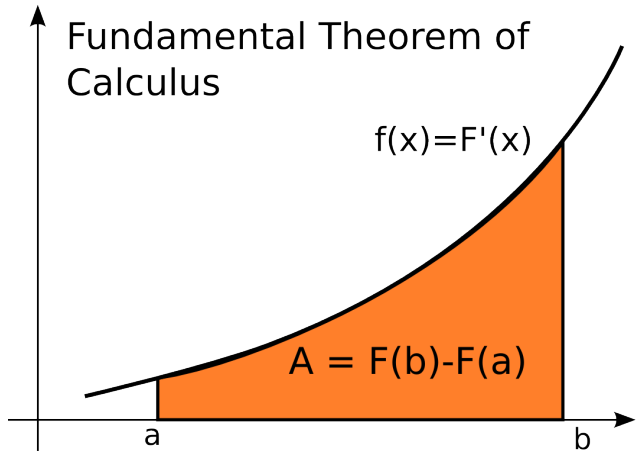
The statement that  $\int_a^b f(x) dx = F(b) - F(a)$  is known as the



## Fundamental Theorem of Calculus.

$$\int_a^b f(x) dx = F(b) - F(a)$$

This statement relates the definite integral  $\int_a^b f(x)dx$  with an antiderivative  $F(x)$  so it relates definite and indefinite integral. Moreover, it provides us with the way of calculating a definite integral *without* computing the sums of any rectangles and finding their limits.



When evaluating the antiderivative  $F(x)$  at  $b$  and  $a$  and subtracting the answers, the notation

$$F(x)|_a^b \text{ is used for } F(b) - F(a).$$

Thus,

to evaluate  $\int_a^b f(x) dx$

1. find an antiderivative  $F(x)$  of  $f(x)$ ,
2. evaluate  $F(x)|_a^b$ ,
3. obtain the numerical answer  $F(b) - F(a)$ .

**Example 1.** Evaluate the integral  $\int_0^2 x^2 dx$  computing the area under  $x^2$  from 0 to 2.

**Solution.** The function  $x^2$  has an antiderivative  $\frac{x^3}{3}$ . Thus

$$\int_0^2 x^2 dx = \frac{x^3}{3} \Big|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} = 2.666\dots$$

Keep in mind the difference between the definite and indefinite integral. If  $f(x)$  is a continuous function with antiderivative  $F(x)$ , then

<u>the indefinite integral</u> $\int f(x) dx$	<u>the definite integral</u> $\int_a^b f(x) dx$
is the family of <b>functions</b> $F(x) + c,$	is the <b>number</b> $F(b) - F(a).$

The fact that the Fundamental Theorem of Calculus enables you to compute the total change in antiderivative of  $f(x)$  when  $x$  changes from  $a$  to  $b$  is referred also as the **Total Change Theorem**. Thus, the definite integral can be used to find the *total change in a quantity on an interval given its rate*.

**Example 2.** Suppose that the velocity of an object is given by the function  $v(t) = 0.3t$  where  $t$  is the time in seconds and  $v$  is the velocity in feet per second. Determine the total movement of the object between 10 and 20 seconds.

**Solution.** The total movement is the total change in position  $s(t)$  from 10 to 30 seconds is  $s(20) - s(10)$ . This can be found as the definite integral from 10 to 20 of the rate  $v(t) = s'(t)$ . Thus,

$$s(20) - s(10) = \int_{10}^{20} v(t) dt = \int_{10}^{20} 0.3t dt = 0.3 \left. \frac{1}{2}t^2 \right|_{10}^{20} = 0.15(20^2 - 10^2) = 0.15(300) = 45 \text{ feet.}$$

**Units of the total change.** If  $[x]$  denotes the units of quantity  $x$  and  $[f(x)]$  denotes the units of  $f(x)$ , then

$$\text{the definite integral } \int_a^b f(x)dx \text{ is in units } [f(x)] \cdot [x].$$

This is because  $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\bar{x}_i)\Delta x$  for some partition of  $[a, b]$  and points  $\bar{x}_i$  from the subinterval  $[x_{i-1}, x_i]$  and the units of the sum on the right are the units  $[f(x)]$  of  $f(\bar{x}_i)$  multiplied by the units  $[x]$  of  $\Delta x$ .

For instance, in Example 2 above, the units of the definite integral are the units of velocity multiplied by the units of time. Thus, the answer is in meters since the product  $\frac{\text{meters}}{\text{second}} \cdot \text{second}$  produces meters.

In some cases you may be instructed to use the calculator if a function is such that finding an antiderivative is out of the scope of Calculus 1 course.

**Example 3.** The size of a certain bacteria culture grows at a rate of  $f(t) = te^{t/2}$  milligrams per hour. Use the Left-Right Sums calculator program to approximate the bacteria size after the first 3 hours to the first two nonzero digits.

**Solutions.** The bacteria size after the first 3 hours can be found as  $\int_0^3 te^{t/2}dt$ . Enter the function in your calculator as  $xe^{x/2}$ , start the program and enter that  $a = 0$ ,  $b = 3$ . With  $n = 300$  for example, both the left and the right sums round to 13. Thus, in 3 hours the size increased by 13 mg.

### Properties of the Definite Integral

Assume that  $f(x)$  and  $g(x)$  are continuous functions on the interval  $[a, b]$  with antiderivatives  $F(x)$  and  $G(x)$  respectively. The Fundamental Theorem of Calculus can be used to show the following properties of definite integrals.

1. Since  $\int_a^a f(x) dx = F(a) - F(a) = 0$ , we have that

$$\int_a^a f(x) dx = 0.$$

2.  $\int_a^b f(x) dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x) dx$  thus we have that

$$\int_a^b f(x) dx = -\int_b^a f(x) dx.$$

3. If  $c$  is any number from  $[a, b]$ , then  $\int_a^b f(x) dx = F(b) - F(a) = F(b) - F(c) + F(c) - F(a) = (F(c) - F(a)) + (F(b) - F(c)) = \int_a^c f(x) dx + \int_c^b f(x) dx$  so that

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

This property can be shown to hold for any number  $c$ , not necessarily between  $a$  and  $b$ .

4. Since  $F(x) + G(x)$  is an antiderivative of  $f(x) + g(x)$  we have that  $\int_a^b (f(x) + g(x)) dx = F(b) + G(b) - (F(a) + G(a)) = F(b) - F(a) + G(b) - G(a) = \int_a^b f(x) dx + \int_a^b g(x) dx$ . Thus

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

5. If  $c$  is any constant,  $cF(x)$  is an antiderivative of  $cf(x)$  so that  $\int_a^b cf(x) dx = cF(b) - cF(a) = c(F(b) - F(a)) = c \int_a^b f(x) dx$ . Thus

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

6. Since the definite integral of a continuous function is not a function but a constant, the variable name used in the definite integral does not matter. In other words,

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du = \dots$$

### Finding the area between $f(x)$ and $x$ -axis on a given interval.

So far we related the definite integral and area under the graph of a continuous function just in the case the function is nonnegative. Recall that in this case the definite integral is exactly *equal* to the area.

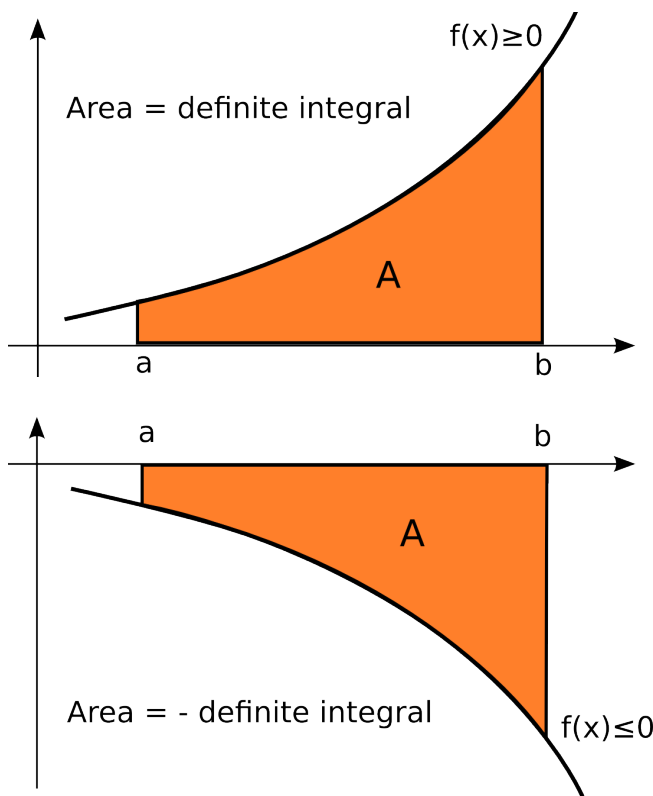
Let us consider now the other computing the area if the function is not nonnegative on entire interval.

First, let us now consider the case when a given function is less or equal to zero on an interval  $[a, b]$ . If  $f(x) \leq 0$  then  $|f(x)| = -f(x) \geq 0$  and the size of the area  $A$  between  $x$ -axis and  $f(x)$  on  $[a, b]$  is the same as the size of the area under the curve of  $|f(x)| = -f(x)$  on  $[a, b]$ . Thus

$$A = \int_a^b -f(x) dx = - \int_a^b f(x) dx$$

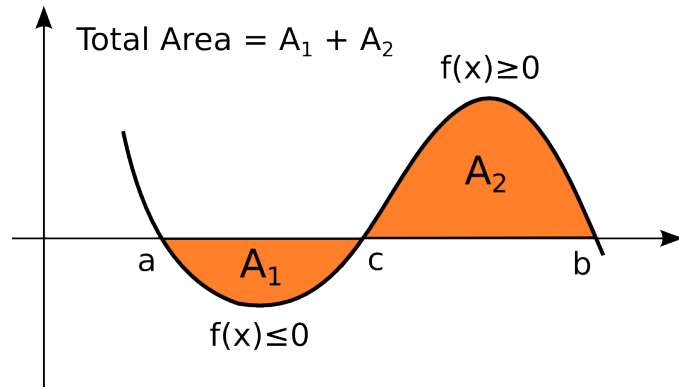
by property 5. above. We can unify the cases  $f(x) \geq 0$  or  $f(x) \leq 0$  by writing that in either case  $\int_a^b |f(x)| dx = \left| \int_a^b f(x) dx \right|$  but note that this does not have to hold if the function  $f(x)$  changes sign on interval  $[a, b]$ .

Let us assume that  $f(x)$  is changing the sign just once at  $c$  in  $[a, b]$ . Say that  $f$  is changing from negative to positive at  $c$  as in the figure below.



On interval  $[a, c]$ ,  $f(x) \leq 0$  so the area  $A_1$  between  $f(x)$  and  $x$ -axis can be found as  $A_1 = -\int_a^c f(x) dx$ . On interval  $[c, b]$ ,  $f(x) \geq 0$  so the area  $A_2$  between  $f(x)$  and  $x$ -axis can be found as  $A_2 = \int_c^b f(x) dx$ . The total area  $A$  can be obtained as the sum  $A_1 + A_2$ . Thus

$$A = A_1 + A_2 = \int_a^c -f(x) dx + \int_c^b f(x) dx.$$



Note that the total area cannot be evaluated using a single definite integral. The property 3. does not apply to the sums of two integrals in the last formula since the integrand in the first integral is  $-f(x)$  and the integrand in the second integral is  $f(x)$ .

Similarly, if  $f$  is changing from positive to negative at  $c$  in  $[a, b]$  we would have that  $A = A_1 + A_2 = \int_a^c f(x) dx + \int_c^b -f(x) dx$ .

If  $f(x)$  has more than one  $x$ -intercept in  $[a, b]$ , one would need to divide the total area in more than two regions.

This brings us to the following procedure for finding the total area under the curve.

**Finding Area.** To find the total area  $A$  between the graph of a continuous function  $f(x)$  and  $x$ -axis on  $[a, b]$ ,

1. Graph the function on  $[a, b]$  and check if  $f$  is positive, negative or it changes the sign.

2. Consider the following cases.

Case 1. If  $f$  is **positive** on  $[a, b]$ , then  $A = \int_a^b f(x) dx$ .

Case 2. If  $f$  is **negative** on  $[a, b]$ , then  $A = \int_a^b -f(x) dx$ .

Case 3. If  $f$  is **changing sign** on  $[a, b]$  proceed with the following steps.

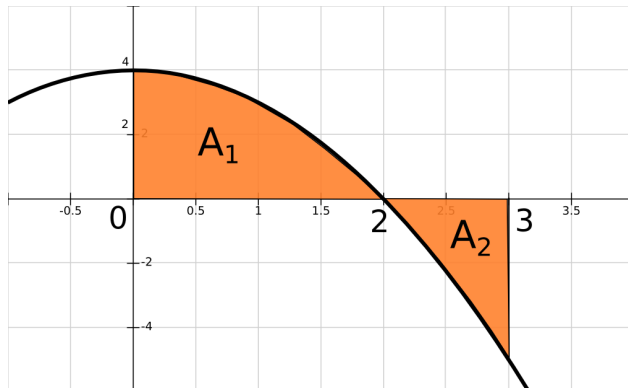
3. Find all  $x$ -intercepts  $c_1, c_2 \dots c_k$  of  $f(x)$  which are in  $[a, b]$  and divide the interval into subintervals such that  $f$  does not change sign on each subinterval.

4. Finding the area between  $f$  and  $x$ -axis on each subinterval falls under either case 1 or 2.

5. The total area  $A$  is the sum of the areas on each subinterval.

**Example 4.** Find the area between the graph of  $f(x) = 4 - x^2$  and  $x$ -axis on  $[0, 3]$ .

**Solution.** Consider the graph of the function on the interval  $[0, 3]$ . Note that function changes the sign from positive to negative at a point between 0 and 3. Set the function to zero to find the  $x$ -intercepts.  $4 - x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ . Since -2 is not in  $[0, 3]$ , just 2 is relevant. Let  $A_1$  denote the area on  $[0, 2]$  and  $A_2$  the area on  $[2, 3]$ .



The function is positive on  $[0,2)$  and negative on  $(2,3]$  so

$$A_1 = \int_0^2 (4 - x^2) dx \quad \text{and} \quad A_2 = \int_2^3 -(4 - x^2) dx = - \int_2^3 (4 - x^2) dx.$$

Find the antiderivative  $4x - \frac{1}{3}x^3$  of  $4 - x^2$  and evaluate the two areas as follows.

$$A_1 = \int_0^2 (4 - x^2) dx = \left(4x - \frac{1}{3}x^3\right) \Big|_0^2 = (4(2) - \frac{1}{3}2^3) - (4(0) - \frac{1}{3}0^3) = 8 - \frac{8}{3} = \frac{16}{3}.$$

$$A_2 = - \int_2^3 (4 - x^2) dx = - \left(4x - \frac{1}{3}x^3\right) \Big|_2^3 = - \left( (4(3) - \frac{1}{3}3^3) - (4(2) - \frac{1}{3}2^3) \right) = -(12 - 9 - 8 + \frac{8}{3}) = \frac{7}{3}.$$

Thus the total area  $A$  is  $A = A_1 + A_2 = \frac{16}{3} + \frac{7}{3} = \frac{23}{3} \approx 7.67$ .

### Practice Problems.

- Evaluate the following. (a)  $\int_0^2 2x \, dx$       (b)  $\int_0^1 x^2 + 2 \, dx$       (c)  $\int_1^4 x^{-2} \, dx$
- Find the area between the graph of  $f(x)$  and  $x$ -axis on  $[a, b]$ .
  - $f(x) = x^2$ ;     $a = 0$ ,     $b = 2$ .
  - $f(x) = x - 3$ ;     $a = 1$ ,     $b = 2$
  - $f(x) = x^2 - 9$ ;     $a = -1$ ,     $b = 4$
  - $f(x) = x^2 - 2x$      $a = 1$ ,     $b = 3$
  - $f(x) = 2\sqrt{x} - 4$ ;     $a = 0$ ,     $b = 9$
- From past records, a botanist knows that a certain species of tree has a rate of growth that can be modeled by  $f(t) = \frac{2}{\sqrt{t}}$ ,  $1 \leq t \leq 4$ , where  $t$  is the age of the tree in years and  $f(t)$  is the growth rate in feet per year. Determine how much did the tree grow from the time when it was a year old to the time it was four years old.
- Suppose that the velocity of an object is given by the function  $v(t) = \frac{t}{\sqrt{t^2+9}}$  where  $t$  is the time in seconds and  $v$  is the velocity in feet per second. Determine the total movement of the object between 3 and 5 seconds.
- The rate of change in the U.S. population can be modeled by  $g(x) = 1.03e^{0.013t}$ ,  $0 \leq t \leq 100$  where  $t$  represents the number of years since 1900 and  $g$  represents the rate of change in population measured in millions per year. Determine the total increase in the U.S. population from 1900 to 1950.
- Geologists estimate that an oil field will produce oil at a rate given by  $f(t) = 600e^{-0.1t}$  thousand barrels per month,  $t$  months into production. Estimate the total production for the first year of operation. Round to the nearest whole number.
- Breathing is cyclic and a full respiratory cycle takes about 5 seconds. The function  $f(t) = \frac{1}{2} \sin \frac{2\pi t}{5}$  in liters per second has often been used to model the rate of air flow into the lungs at time  $t$ . Find the volume of inhaled air in the lungs in one respiratory cycle.

### Solutions.

1. (a)  $\int_0^2 2x \, dx = 2 \frac{x^2}{2} \Big|_0^2 = x^2 \Big|_0^2 = 4 - 0 = 4.$

(b)  $\int_0^1 x^2 + 2 \, dx = \left( \frac{x^3}{3} + 2x \right) \Big|_0^1 = \frac{1^3}{3} + 2(1) - 0 = \frac{7}{3}.$

(c)  $\int_1^4 x^{-2} \, dx = \frac{x^{-1}}{-1} \Big|_1^4 = \frac{-1}{x} \Big|_1^4 = \frac{-1}{4} - \frac{-1}{1} = \frac{3}{4}.$

2. (a) Since  $f(x)$  is positive on the given interval, the area is  $A = \int_0^2 x^2 dx = \frac{x^3}{3} \Big|_0^2 = \frac{8}{3}.$

(b) Graph the function  $f(x) = x - 3$  and note that it is negative on  $[1,2]$ . Thus, the area is  $A = -\int_1^2 (x - 3) dx = -\left( \frac{x^2}{2} - 3x \right) \Big|_1^2 = -2 + 6 + \frac{1}{2} - 3 = 1 + \frac{1}{2} = \frac{3}{2}.$

(c) Graph the function and check the sign of  $f(x)$  on  $[-1,4]$ . Notice that  $f(x)$  changes the sign. Find  $x$ -intercepts:  $f(x) = x^2 - 9 = 0$  when  $x^2 = 9 \Rightarrow x = \pm 3$ . The relevant zero is 3 since -3 is not in the interval  $[-1,4]$ . From the graph, you can see that  $f(x)$  is negative on  $[-1, 3]$  and positive on  $(3, 4]$ . Thus, the total area can be obtained as the sum of  $A_1 = -\int_{-1}^3 f(x) dx$  and  $A_2 = \int_3^4 f(x) dx$ .

$$A_1 = -\int_{-1}^3 (x^2 - 9) dx = \left( -\frac{x^3}{3} + 9x \right) \Big|_{-1}^3 = -9 + 27 - \frac{1}{3} + 9 = \frac{80}{3} = 26.67. \quad A_2 = \int_3^4 (x^2 - 9) dx = \left( \frac{x^3}{3} - 9x \right) \Big|_3^4 = \frac{64}{3} - 36 - 9 + 27 = \frac{10}{3} = 3.33$$

The total area is  $A = A_1 + A_2 = \frac{80}{3} + \frac{10}{3} = 30.$

**Careful:** the total area is *not*  $\int_{-1}^4 f(x) dx = -23.33.$

(d) Check the sign of  $f(x)$  on  $[1,3]$  and note that  $f(x)$  changes the sign. Find  $x$ -intercepts:  $f(x) = x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0, x = 2$ . The relevant zero is 2 since 0 is not in the interval  $[1,3]$ . From the graph, you can see that  $f(x)$  is negative on  $[1, 2)$  and positive on  $(2, 3]$ . Thus, the total area can be obtained as the sum of  $A_1 = -\int_1^2 f(x) dx$  and  $A_2 = \int_2^3 f(x) dx$ .

$$A_1 = -\int_1^2 (x^2 - 2x) dx = \left( -\frac{x^3}{3} + x^2 \right) \Big|_1^2 = -\frac{8}{3} + 4 + \frac{1}{3} - 1 = \frac{2}{3} = 0.67. \quad A_2 = \int_2^3 (x^2 - 2x) dx = \left( \frac{x^3}{3} - x^2 \right) \Big|_2^3 = 9 - 9 - \frac{8}{3} + 4 = \frac{4}{3} = 1.33$$

The total area is  $A = A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = 2.$

**Careful:** the total area is *not*  $\int_1^3 f(x) dx = 0.67.$

(e) Check the sign of  $f(x)$  on  $[0,9]$  and note that  $f(x)$  changes the sign. Find  $x$ -intercepts  $f(x) = 2\sqrt{x} - 4 = 0$  when  $2\sqrt{x} = 4 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$ . From the graph, you can see that  $f(x)$  is negative on  $[0, 4)$  and positive on  $(4, 9]$ . Thus, the total area can be obtained as the sum of the area  $A_1 = -\int_0^4 f(x) dx$  and  $A_2 = \int_4^9 f(x) dx$ . Similarly as in previous problems, you can calculate that  $A_1 = \frac{16}{3} = 5.33$  and  $A_2 = \frac{16}{3} = 5.33$ . The total area is  $A = A_1 + A_2 = \frac{32}{3} = 10.67.$

**Careful:** the total area is *not*  $\int_0^9 f(x) dx = 0.$

3. The total growth can be found by integrating the growth rate from 1 to 4.  $\int_1^4 \frac{2}{\sqrt{t}} dt = 2 \frac{t^{1/2}}{1/2} \Big|_1^4 = 4\sqrt{t} \Big|_1^4 = 8 - 4 = 4$  feet.

4. If  $s(t)$  denotes the distance traveled, the problem is asking for  $s(5) - s(3) = \int_3^5 v(t) dt$ . Using the substitution with  $u = t^2 + 9$  find the antiderivative of the velocity function to be  $\sqrt{t^2 + 9}$ . Thus  $s(5) - s(3) = \sqrt{t^2 + 9} \Big|_3^5 = \sqrt{34} - \sqrt{18} \approx 1.588$  ft.

5. The total increase in the number of people from  $t = 0$  to  $t = 5$  can be found as  $\int_0^{50} 1.03e^{0.013t} dt$ . Use the substitution  $u = 0.013t$  to find the antiderivative and get  $1.03 \frac{1}{0.013} e^{0.013t} \Big|_0^{50} = \frac{1.03}{0.013} (e^{0.013(50)} - 1) \approx 79.23(0.916) = 72.54$  millions.
6. Work out the details similarly to the previous problem. The total production for the first year of operation can be estimated to be  $4192.834 \approx 4193$  thousands of barrels.
7. Graph the function to note that the inhaling occurs in the first 2.5 seconds of the cycle and exhaling in the second 2.5 seconds. Thus, the volume of inhaled air can be found as  $\int_0^{5/2} \frac{1}{2} \sin \frac{2\pi t}{5} dt$ . Using the substitution  $u = \frac{2\pi t}{5}$ , you obtain  $\frac{1}{2} \frac{-5}{2\pi} \cos \frac{2\pi t}{5} \Big|_0^{5/2} = \frac{-5}{4\pi} (-1 - 1) = \frac{5}{2\pi} \approx 0.796$  liters.