Calculus 1
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Implicit Differentiation. Logarithmic Differentiation

Implicit functions. If a function can be expressed as $y = f(x)$ it is said to be in the **explicit** form. However, in some cases, the variables x and y can be related with an equation $F(x, y) = 0$ which cannot be solved for y. The relation

$$F(x, y) = 0$$

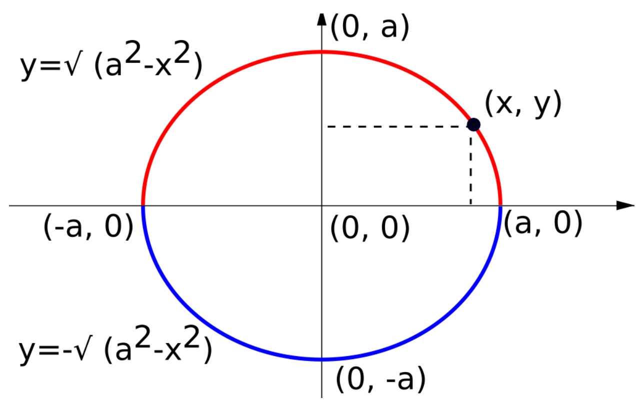
is said to define an **implicit function**. If a function is given in the form $f(x, y) = g(x, y)$ one can always consider it in the form $F(x, y) = 0$ when subtracting the right hand side from the whole equation.

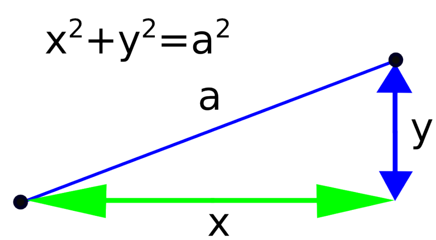
The graph of the relation $F(x, y) = 0$ may fail the vertical line and so it may not be a function but a collection of several pieces each of which passes the vertical line test and, for itself constitutes a function. Despite of this misnomer, the curve defined by $F(x, y) = 0$ is still refer to as an implicit *function*.

The most famous example. Probably the most widely encountered example of an implicit function is a circle. Recall that the equation of a circle of radius a centered at the origin is

$$x^2 + y^2 = a^2.$$

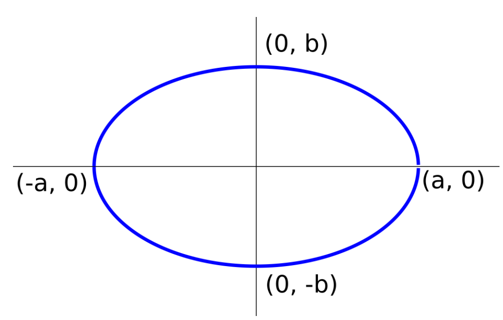
The circle fails the vertical line test so it is not a function. When solving for you, you have that $y^2 = a^2 - x^2$ and so $y = \pm\sqrt{a^2 - x^2}$. This also indicates that the circle cannot be described with a single explicit function. The upper half is given by $y = \sqrt{a^2 - x^2}$ and the lower half by $y = -\sqrt{a^2 - x^2}$. The fact that the circle cannot be described by a single function greatly contributes to introduction of parametric and polar coordinates which you will cover in Calculus 2.





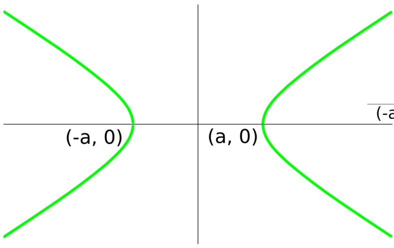
Other examples of implicit curves include:

- ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,
- hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,
- lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$,
- astroid $x^{2/3} + y^{2/3} = a^{2/3}$, and
- the folium of Descartes $x^3 + y^3 = 3axy$,

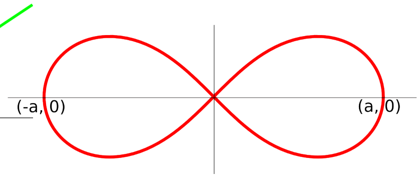


Ellipse

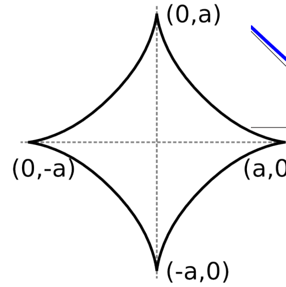
where a and b are positive constants. Watch the animations on Wikipedia to understand the origin and relevance of these curves.



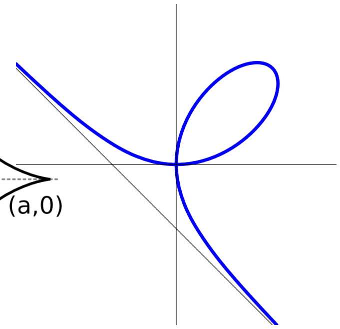
Hyperbola,



lemniscate,



astroid, and



the folium of Descartes

Derivative of an implicit function. It is possible to find the derivative of an implicit function $F(x, y) = 0$ even without solving the equation for y to do that, one can

1. Differentiate the whole equation $F(x, y) = 0$
   using the chain rule for differentiation the terms containing y.

2. Solve for the derivative y'.

Let us illustrate this method on finding derivative of the circle centered at the origin.

Example 1. Find the derivative y' of the circle $x^2 + y^2 = 5$ and use it to find an equation of the tangent line at the point $(2, 1)$.

Solution. Start by differentiating the equation $x^2 + y^2 = 5$.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}5 \quad \Rightarrow \quad \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0 \quad \Rightarrow \quad 2x + 2y \cdot y' = 0$$

Note that we used the chain rule when differentiating y^2. Treat the function y^2 as the composite of the outer y^2 and the inner y. The derivative of the outer is $2y$ and the derivative of the inner is y'. So, the derivative is $2y \cdot y'$.

Finally, solve the last equation for y' to get

$$2x + 2yy' = 0 \quad \Rightarrow \quad 2yy' = -2x \quad \Rightarrow \quad yy' = -x \quad \Rightarrow \quad y' = \frac{-x}{y}.$$

At the point $(2, 1)$ the derivative has value $y' = \frac{-x}{y} = \frac{-2}{1} = -2$ giving you the slope of the tangent line. So, the tangent line is $y - 1 = -2(x - 2) \Rightarrow y = -2x + 5$.

Example 2. Find the derivative of the function $(3x + 2y)^3 = ye^{2x} + 7$ and evaluate it at $(0, 1)$.

Solution. Differentiate both side of the equation. Use the chain rule for the left side with inner function $3x + 2y$. The derivative of this inner function is $3 + 2y'$. Use the product rule for the right hand side with $f(x) = y$ and $g(x) = e^{2x}$ so that $f'(x) = y'$ and $g'(x) = 2e^{2x}$. Thus we have

$$\frac{d}{dx}(3x + 2y)^3 = \frac{d}{dx}(ye^{2x}) + \frac{d}{dx}(7) \quad \Rightarrow \quad 3(3x + 2y)^2(3 + 2y') = y'e^{2x} + 2ye^{2x}$$

In order to solve for y', you need to keep all the terms with y' on the same side. Distribute the terms on the left side so that it is clear that it consists of two terms, one with and one without y'

$$9(3x+2y)^2 + 6y'(3x+2y)^2 = y'e^{2x} + 2ye^{2x}$$

Group the two terms with y' to the same side and the remaining two terms on the other side.

$$6y'(3x+2y)^2 - y'e^{2x} = 2ye^{2x} - 9(3x+2y)^2$$

Finally, factor y' on the left side and solve for it.

Note that the left side consists of two terms, the first

$$y'\left(6(3x+2y)^2 - e^{2x}\right) = 2ye^{2x} - 9(3x+2y)^2 \quad \Rightarrow \quad y' = \frac{2ye^{2x} - 9(3x+2y)^2}{6(3x+2y)^2 - e^{2x}}$$

When $x = 0$ and $y = 1$, $y' = \frac{2(1)e^0 - 9(0+2)^2}{6(0+2)^2 - 1} = \frac{2-36}{24-1} = \frac{-34}{23} \approx 1.48$.

Example 3. Using implicit differentiation and the fact that derivative of $\sin x$ is $\cos x$, show that the derivative of $\cos x$ is $-\sin x$.

Solution. Start from the identity $\sin^2 x + \cos^2 x = 1$ which relates the sine and cosine function and differentiate both sides.

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}(1) \Rightarrow 2\sin x \frac{d}{dx}(\sin x) + 2\cos x \frac{d}{dx}(\cos x) = 0 \Rightarrow$$

$$2\sin x \cos x + 2\cos x \frac{d}{dx}(\cos x) = 0 \Rightarrow \sin x + \frac{d}{dx}(\cos x) = 0 \Rightarrow \frac{d}{dx}(\cos x) = -\sin x.$$

Logarithmic Differentiation.

Assume that the function has the form $y = f(x)^{g(x)}$ where both f and g are non-constant functions. Although this function is not implicit, it does not fall under any of the forms for which we developed differentiation formulas so far. This is because of the following.

- In order to use the power rule, the exponent needs to be constant.
- In order to use the exponential function differentiation formula, the base needs to be constant.

Thus, no differentiation rule covers the case $y = f(x)^{g(x)}$. These functions sill can be differentiated by using the method known as the **logarithmic differentiation.**

To differentiate a function of the form $y = f(x)^{g(x)}$ follow the steps of the logarithmic differentiation below.

1. Take ln of both sides of the equation $y = f(x)^{g(x)}$.
2. Rewrite the right side $\ln f(x)^{g(x)}$ as $g(x) \cdot \ln(f(x))$.
3. Differentiate both sides.
4. Solve the resulting equation for y'.

Example 4. Find the derivative of $y = x^x$.

Solution. Follow the steps of the logarithmic differentiation.

1. First take ln of each side to get $\ln y = \ln x^x$.
2. Rewrite the right side as $x \ln x$ to get $\ln y = x \ln x$.
3. Then differentiate both sides. Use the chain rule for the left side noting that the derivative of the inner function y is y'. Use the product rule for the right side. Obtain $\frac{1}{y}y' = \ln x + \frac{1}{x}\,x$.
4. Multiply both sides with y to solve for y' and get $y' = (\ln x + 1)y$. Finally, recall that $y = x^x$ to get the derivative solely in terms of x as
$$y' = (\ln x + 1)x^x.$$

Practice problems.

1. Find the derivative $\frac{dy}{dx}$ of the following implicit functions.

   (a) $x^2 + xy^4 = 6$ (b) $x^3 + 12xy = y^3$ (c) $xe^y + x^2 = y^2$

2. Find an equation of the line tangent to the graph of the given curves at the indicated point.

   (a) $x^2 + y^2 = 13;\;\; (3, 2)$ (b) $x \ln y = 2x^3 - 2y;\;\; (1, 1)$ (c) $x^2 + y^2 = e^y;\;\; (1, 0)$

3. Find the derivative of the following functions.

   (a) $y = (3x)^{5x}$ (b) $y = x^{\ln x}$ (c) $y = (\ln x)^x$ (d) $y = (3x + 2)^{2x-1}$

Solutions.

1. (a) $\frac{d}{dx}(x^2 + xy^4) = \frac{d}{dx}(6) \Rightarrow 2x + 1 \cdot y^4 + 4y^3 \cdot y' \cdot x = 0 \Rightarrow 4y^3xy' = -2x - y^4 \Rightarrow y' = \frac{-2x-y^4}{4xy^3} = -\frac{2x+y^4}{4xy^3}$.

   (b) $\frac{d}{dx}(x^3 + 12xy) = \frac{d}{dx}(y^3) \Rightarrow 3x^2 + 12 \cdot y + y' \cdot 12x = 3y^2 \cdot y' \Rightarrow 3x^2 + 12y = 3y^2y' - 12xy' \Rightarrow 3x^2 + 12y = (3y^2 - 12x)y'$ $y' = \frac{3x^2+12y}{3y^2-12x} = \frac{x^2+4y}{y^2-4x}$.

   (c) $\frac{d}{dx}(xe^y + x^2) = \frac{d}{dx}(y^2) \Rightarrow 1 \cdot e^y + e^y y' \cdot x + 2x = 2y \cdot y' \Rightarrow e^y + 2x = 2yy' - xy'e^y \Rightarrow e^y + 2x = (2y - xe^y)y'$ $y' = \frac{e^y+2x}{2y-xe^y}$.

2. (a) Find the derivative first using implicit differentiation. $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(13) \Rightarrow 2x + 2y \cdot y' = 0 \Rightarrow 2yy' = -2x \Rightarrow y' = \frac{-x}{y}$. At point $(3, 2)$, the derivative is $y' = \frac{-3}{2}$. So, the tangent line is $y - 2 = \frac{-3}{2}(x - 3) \Rightarrow y = \frac{-3}{2}x + \frac{13}{2}$.

   (b) $\frac{d}{dx}(x \ln y) = \frac{d}{dx}(2x^3 - 2y) \Rightarrow \ln y + \frac{1}{y} \cdot y' \cdot x = 6x^2 - 2y' \Rightarrow 2y' + \frac{x}{y}y' = 6x^2 - \ln y \Rightarrow (2 + \frac{x}{y})y' = 6x^2 - \ln y \Rightarrow y' = \frac{6x^2-\ln y}{2+\frac{x}{y}}$. At point $(1, 1)$, the derivative is $y' = \frac{6-0}{2+1} = 2$. The tangent line is $y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$.

   (c) $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(e^y) \Rightarrow 2x+2y \cdot y' = e^y y' \Rightarrow 2x = e^y y' - 2yy' \Rightarrow 2x = (e^y - 2y)y' \Rightarrow y' = \frac{2x}{e^y-2y}$. At point $(1, 0)$, the derivative is $y' = \frac{2}{1-0} = 2$. The tangent line is $y - 0 = 2(x - 1) \Rightarrow y = 2x - 2$.

3. (a) Use logarithmic differentiation $\ln y = \ln(3x)^{5x} = 5x\ln(3x) \Rightarrow \frac{1}{y}y' = 5\ln(3x) + \frac{3}{3x}5x \Rightarrow y' = (5\ln(3x)+5)y \Rightarrow y' = (5\ln(3x)+5)(3x)^{5x}$.

(b) Use logarithmic differentiation $y = x^{\ln x} \Rightarrow \ln y = \ln x^{\ln x} = \ln x \ln x = (\ln x)^2 \Rightarrow \frac{1}{y}y' = 2\ln x\frac{1}{x} \Rightarrow y' = (2\ln x\frac{1}{x})x^{\ln x}$.

(c) Use logarithmic differentiation $\ln y = \ln(\ln x)^x = x\ln(\ln x) \Rightarrow \frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x}\frac{1}{x}x \Rightarrow y' = \left(\ln(\ln x) + \frac{1}{\ln x}\right)y \Rightarrow y' = (\ln(\ln x) + \frac{1}{\ln x})(\ln x)^x$.

(d) Use logarithmic differentiation $\ln y = \ln(3x+2)^{2x-1} = (2x-1)\ln(3x+1) \Rightarrow \frac{1}{y}y' = 2\ln(3x+1) + \frac{3(2x-1)}{3x+2} \Rightarrow y' = \left(2\ln(3x+1) + \frac{3(2x-1)}{3x+2}\right)y \Rightarrow y' = (2\ln(3x+2) + \frac{3(2x-1)}{3x+2})(3x+2)^{2x-1}$.