Calculus 1
Lia Vas

Increasing/Decreasing Test. Extreme Values and The First Derivative Test.

Recall that a function $f(x)$ is **increasing** on an interval if the increase in x-values implies an increase in y-values for all x-values from that interval.

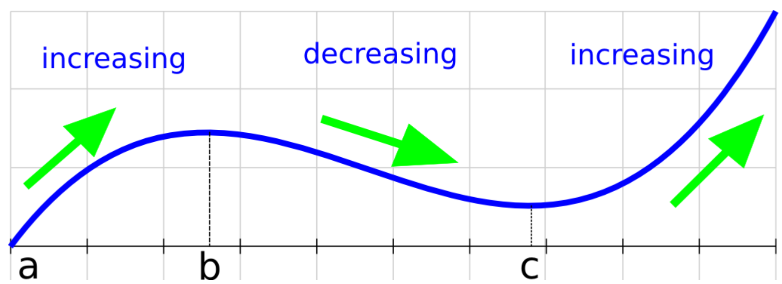
$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2).$$

Analogously, $f(x)$ is **decreasing** if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2).$$

These concepts completely correspond to the intuitive "going up" and "going down" when looking at the graph. One just needs to keep in mind the positive direction of the x-axis when considering the "direction" of the curve.

Example 1. Determine the intervals on which the function with the graph on the right defined on interval (a, ∞) is increasing/decreasing.



Solution. The function is increasing on intervals (a, b) and (c, ∞) and decreasing on (b, c).

The slope of the line tangent to an increasing function at a point is positive. Thus, increasing differentiable functions have positive derivatives. The converse is also true: if a derivative of a differential function is positive for all x-values from an interval, then the function is increasing on that interval. These two statements combine in the following equivalence [1] and the analogous equivalence holds for decreasing functions as well.

> **Increasing/Decreasing Test.** A differentiable function $f(x)$ is:
>
> - increasing on an interval if and only if the derivative $f'(x)$ is positive on that interval,
> - decreasing on an interval if and only if the derivative $f'(x)$ is negative on that interval.

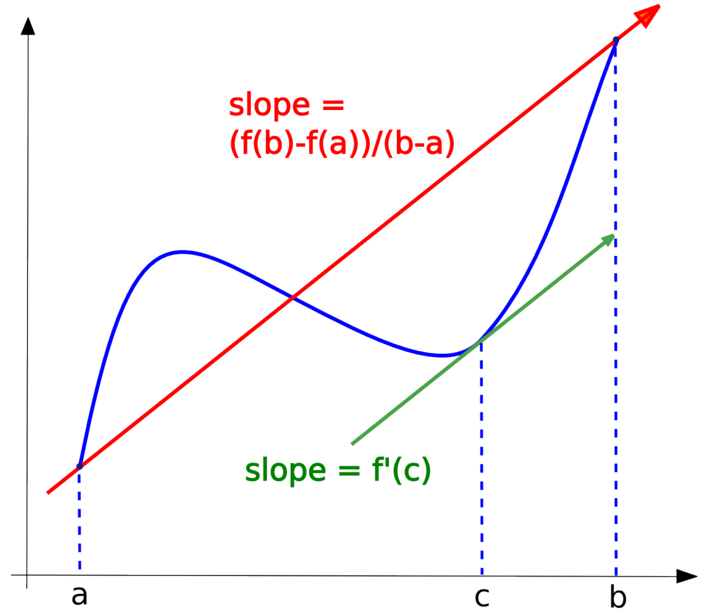
The proof of the converse follows from another statement known as the **Mean Value Theorem**. This theorem states the following.

[1] If a statement p implies a statement q (written as $p \Rightarrow q$) and the converse $q \Rightarrow p$ is also true, we say that p holds **if and only if** q holds and write $p \Leftrightarrow q$. The statement of the form $p \Leftrightarrow q$ is known as an **equivalence**. For example, the statements $x + 3 = 8$ and $x = 5$ are equivalent.

Mean Value Theorem.
If a function $f(x)$ is differentiable on $[a, b]$ then there is c in (a, b) such that

$f(b) - f(a) = f'(c)(b - a).$

This theorem is stating that the slope of the secant line passing $(a, f(a))$ and $(b, f(b))$ is the same as the slope of the line tangent to $f(x)$ at some point from the interval. Graphing differentiable functions just as on the figure on the right, you can convince yourself that this claim is satisfied in all the examples.



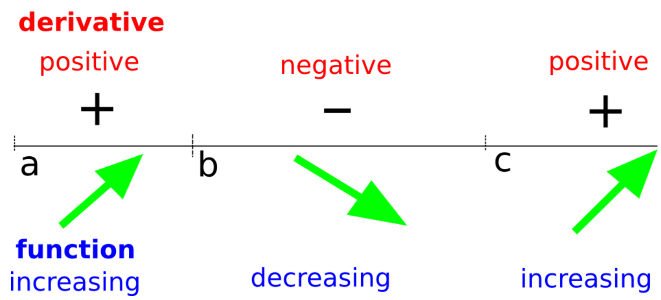
We outline the proof of this theorem in the footnote below [2]. You can fill the blanks from the external sources (for example Wikipedia).

To prove the Increasing/Decreasing Test, consider two points x_1 and x_2 from an interval such that $x_1 < x_2$. By the Mean Value Theorem, there is c from (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1).$$

Since $f'(c) > 0$ and $x_2 - x_1 > 0$ by assumption, we have that $f(x_2) - f(x_1) > 0$. Thus, we have shown that $x_1 < x_2$ implies that $f(x_1) < f(x_2)$ so that the function is increasing.

Thus, determining the intervals of increase/decrease can be done by analyzing the **sign of the derivative**. Recall the number line test method from Precalculus for finding the intervals on which a function is positive/negative. For example, the number line test applied to the derivative of the function from Example 1, would be as follows.



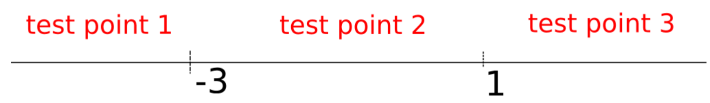
Example 2. Determine the intervals on which the following functions are increasing/decreasing.

(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2 + 4}{2x}$

Solutions. (a) Find the derivative $f'(x) = 3x^2 + 6x - 9$ and factor it as $f'(x) = 3(x^2 + 2x - 3) = 3(x - 1)(x + 3)$. This tells you that the derivative can change sign when $x = 1$ and $x = -3$ so there are the relevant points to consider on the number line.

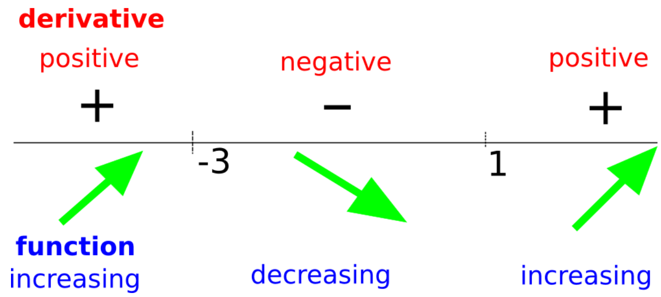
[2] One of few possible proofs of the Mean Value Theorem requires another intuitively clear statement know as the Rolle's Theorem (claiming that a function $f(x)$ differentiable on $[a, b]$ with $f(a) = f(b)$ is such that $f'(c) = 0$ for some c in (a, b)). Rolle's Theorem, on the other hand, is proven using yet another intuitively clear statement know as the Extreme Value Theorem (claiming that a continuous function attains the maximum and minimum on a closed interval). The Mean Value Theorem follows from Rolle's theorem by considering the function which is the difference of $f(x)$ and the line passing $(a, f(a))$ with the slope $\frac{f(b)-f(a)}{b-a}$. The details of the proof can be found online (e.g. on Wikipedia).

Recall that in the precalculus course you may have used the test points from each of the subintervals of the number line. For example, the points -3 and 1 divide the number line to three parts. Take a test point from each part.



For example, with $-4, 0$ and 2 as the test points, you obtain that $f'(-4) = 15 > 0$, $f'(0) = -9 < 0$ and $f'(2) = 15 > 0$. *Careful:* you want to plug the test points in the derivative f' not in the function f since you are determining the sign of f', not f.

Thus we have the number line on the right.

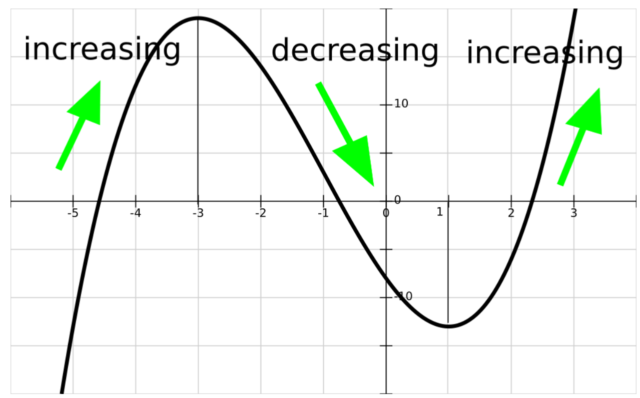


From the number line, we conclude that $f(x)$ is increasing for $x < -3$ and $x > 1$ and decreasing for $-3 < x < 1$. Alternatively, you can write your answer using interval notation as follows.

$f(x)$ is increasing on $(-\infty, -3)$ and $(1, \infty)$,

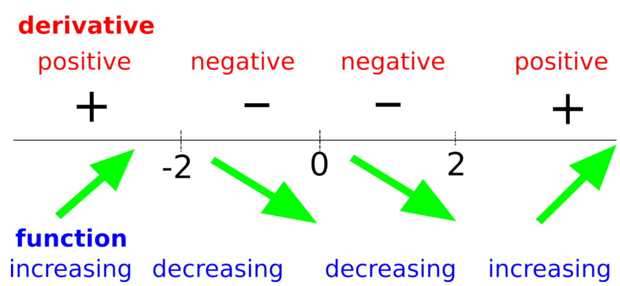
$f(x)$ is decreasing on $(-3, 1)$.

Finally, graph the function and make sure that the graph agrees with your findings.



(b) Use the quotient rule to find the derivative $f'(x) = \frac{2x(2x)-2(x^2+4)}{(2x)^2} = \frac{4x^2-2x^2-8}{4x^2} = \frac{2x^2-8}{4x^2} = \frac{x^2-4}{2x^2} = \frac{(x-2)(x+2)}{2x^2}$.

Three terms impact the sign of $f'(x) : x - 2, x + 2$ and $2x^2$. These terms change the sign for $x = 2, -2$ and 0 so those are the numbers relevant for the number line. Since these three values divide the number line to four pieces, you need four test points. Test the points and obtain the number line on the right.

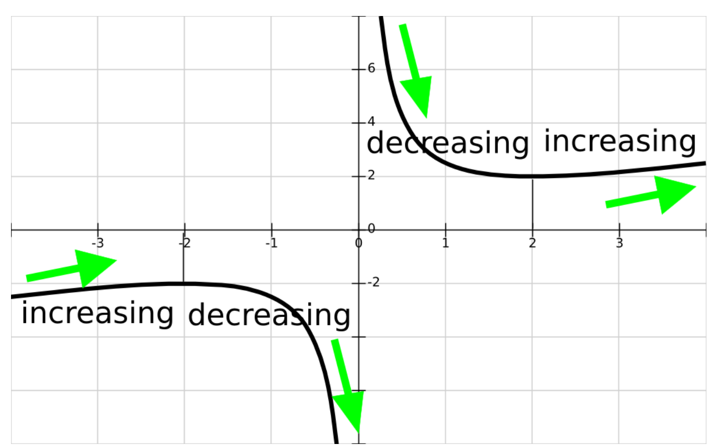


From the number line, we conclude that $f(x)$ is increasing for $x < -2$ and $x > 2$ and decreasing for $-2 < x < 0$ and $0 < x < 2$. Alternatively, using interval notation you have that.

$f(x)$ is increasing on $(-\infty, -2)$ and $(2, \infty)$,

$f(x)$ is decreasing on $(-2, 0)$ and $(0, 2)$.

Finally, graph the function and make sure that the graph agrees with your findings.

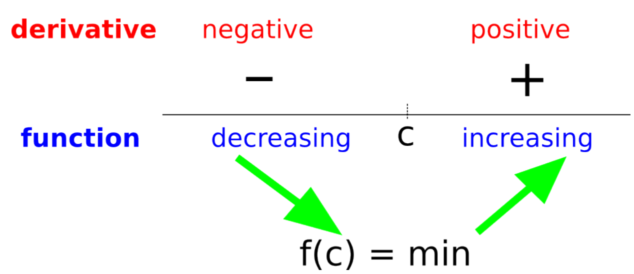


We have seen that the positive/negative sign of the first derivative corresponds exactly to function increasing/decreasing. Next we consider what happens at the points at which the derivative is neither positive, nor negative. At such points the derivative is either zero or undefined and are called the critical points.

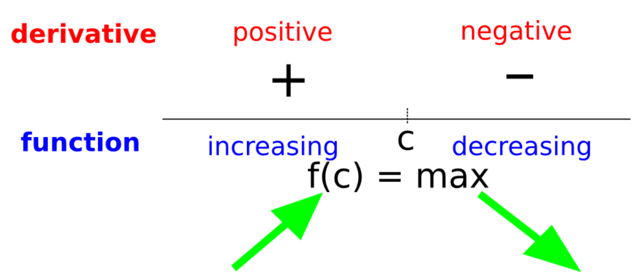
> The point $x = c$ is a **critical point** of a function $f(x)$ if
>
> $f'(c) = 0$ or $f'(c)$ does not exist.

If f is a continuous function, and c is a critical point of f, the following three scenarios are possible.

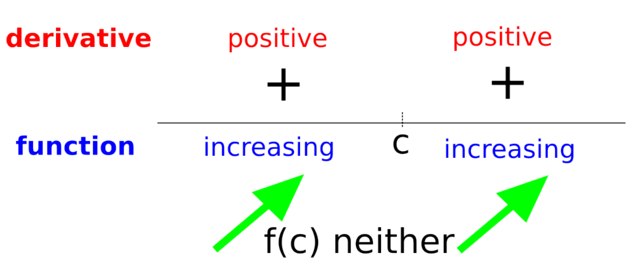
Case 1 The sign of f' is changing from negative to positive. This means that the function is decreasing before c, then reaches the bottom at $x = c$, and then increases after c. So, if defined, $f(c)$ is smaller than all values $f(x)$ when x is in some interval around c. In this case, the function f is said to have a **relative minimum** at $x = c$ and the y-value $f(c)$ is called the **minimal value.**

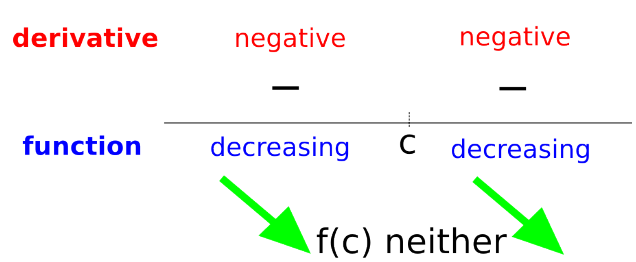


Case 2 The sign of f' is changing from positive to negative. This means that the function is increasing before c, then reaches the top at $x = c$, and then decreases after c. So, if defined, $f(c)$ is larger than all values $f(x)$ when x is in some interval around c. In this case, the function f is said to have a **relative maximum** at $x = c$ and the y-value $f(c)$ is called the **maximal value.**



Case 3 The sign of f' is not changing at $x = c$ (it is either positive both before or after c or negative both before or after c). In this case, f has neither minimum nor maximum at $x = c$.





The minimum and maximum values are collectively referred to as the **extreme values**.

The existence of the third case demonstrates that *a function does not necessarily have a minimum or maximum value at a critical value.*

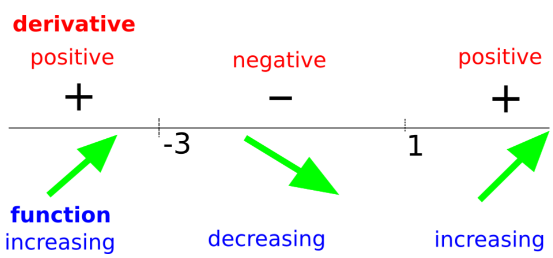
Using the number line test just as when determining increasing/decreasing intervals, one can readily classify the critical points into three categories matching the three cases above and determine the points at which a function has extreme values. This procedure is known as the **First Derivative Test.** Let us summarize.

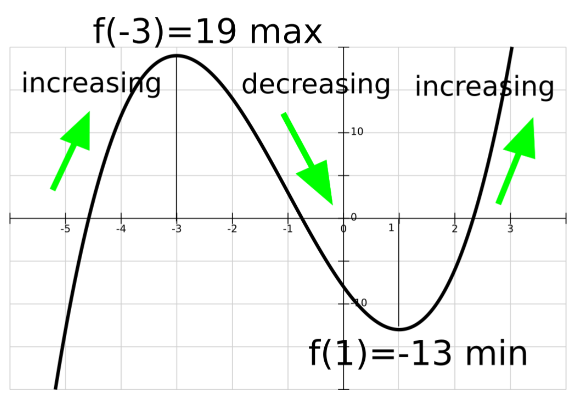
The First Derivative Test. To determine the extreme values of a continuous function $f(x)$:
1. Find $f'(x)$.
2. Set it to zero and find all the critical points.
3. Use the number line to classify the critical points into the three cases.
    - if $f'(x)$ changes from negative to positive at c and $f(c)$ is defined, then f has a minimum at c,
    - if $f'(x)$ changes from positive to negative at c and $f(c)$ is defined, then f has a maximum at c,
    - if $f'(x)$ does not change the sign at c, f has no extreme value c.

Example 2 revisited. Determine the extreme values of the following functions.

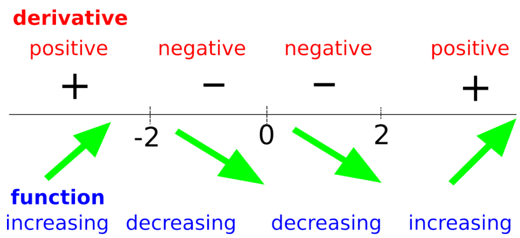
(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$

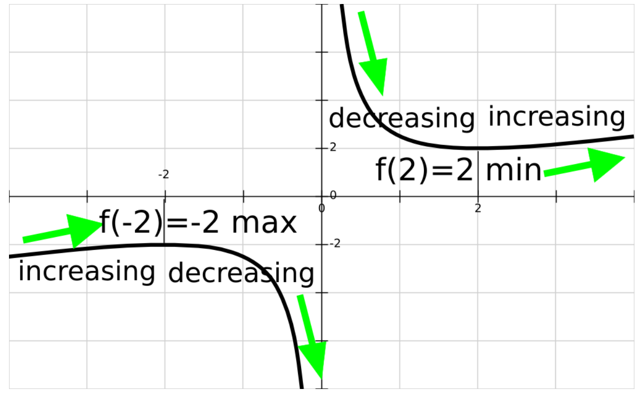
Solutions. (a) Recall that the derivative is $f'(x) = 3(x^2 + 2x - 3) = 3(x-1)(x+3)$ so that $x = 1$ and $x = -3$ are the critical points. In Example 2, we have performed the number line test for the derivative and obtained the number line below. Thus, both critical values are extreme values and there is a minimum at $x = 1$ and a maximum at $x = -3$. Compute the y-values to determine the minimal value $f(1) = -13$ and the maximal value $f(-3) = 19$. Consider the graph again to make sure that the graph agrees with your findings.





(b) Recall that the derivative is $f'(x) = \frac{(x-2)(x+2)}{2x^2}$. Thus the critical points are $x = 2, x = -2$ (at which f' is zero) and $x = 0$ at which f' is not defined. In Example 2, we have performed the number line test for the derivative and obtained the number line below. Thus, both 2 and -2 are extreme values and there is a minimum at $x = 2$ and a maximum at $x = -2$. The derivative is not changing sign at 0 so it is not an extreme value. f is also undefined at 0 so it cannot have an extreme value at 0 for that reason too. Compute the y-values to determine the minimal value $f(2) = 2$ and the maximal value $f(-2) = -2$. Consider the graph again to make sure that the graph agrees with your findings.





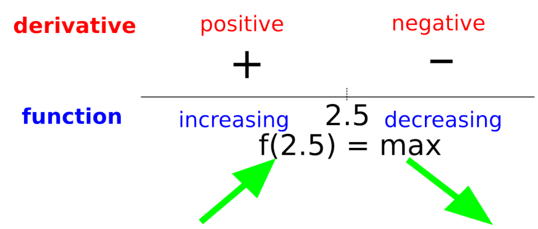
Example 3. A **drug concentration function** is any function describing the concentration C (in μg/cm^3) of a drug in the body at time t hours after the drug was administered. Consider the drug concentration function

$$C(t) = 2te^{-.4t}.$$

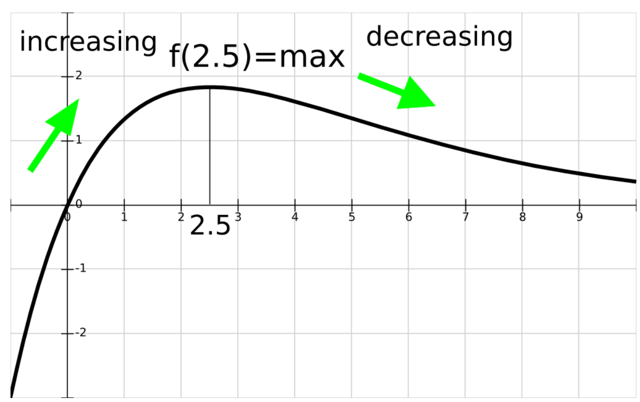
Determine the time intervals when the drug concentration is increasing/decreasing. Determine also the maximal concentration and the time when it is reached. Note that the negative time values are not relevant.

Solution. Use the product rule to find the derivative $C'(t) = 2e^{-.4t} + e^{-.4t}(-.4)2t = 2e^{-.4t}\,(1 - .4t)\,.$ Set the derivative to zero. Since the exponential function is always positive (alternatively $e^{-.4t} = 0 \Leftrightarrow -.4t = \ln 0$ which is not defined), the only critical point comes by solving $1 - .4t = 0$. Thus $t = \frac{1}{.4} = \frac{5}{2} = 2.5$ is the only critical point.

Perform the number line test and obtain the findings on the figure. From the number line, we conclude that the function is increasing on $(-\infty, 2.5)$ and decreasing on $(2.5, \infty)$. At the critical point 2.5 the function reaches its maximum of $f(2.5) = 5e^{-1} \approx 1.84$.



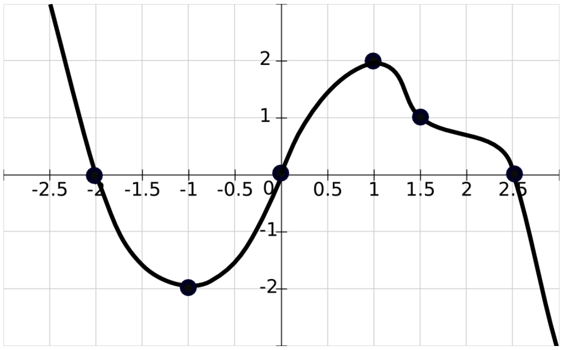
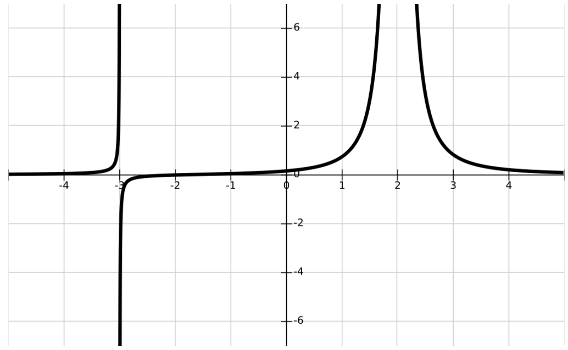
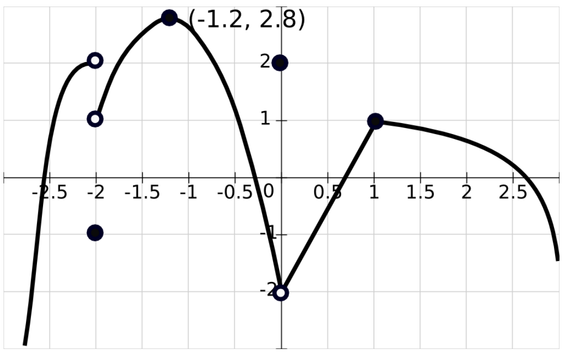
Graph the function to make sure that the graph agrees with your findings.

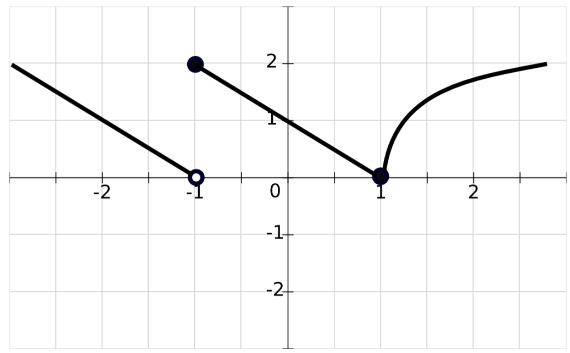


Finally, interpret the conclusions in context of the problem. Since the negative time values are not relevant, we can consider the interval (0, 2.5) instead of $(-\infty, 2.5)$ and conclude that the drug concentration is increasing in the first 2.5 hours, at 2.5 hours it reaches a max value of 1.84 μg/cm^3 and after 2.5 hours the concentration is decreasing.

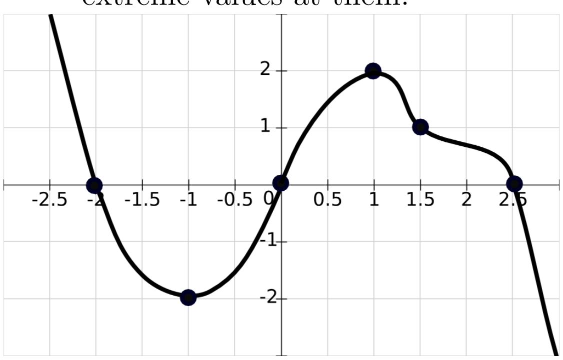
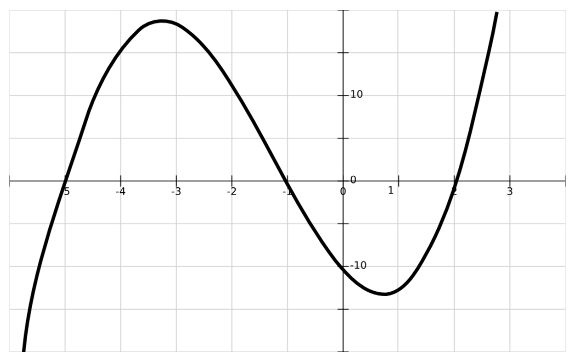
Practice Problems.

1. Find the intervals where the following functions given by their graphs are increasing/decreasing. Determine the critical points and the relative minimum and maximum values (if any).



2. Assume that the graphs below are graphs of *derivative* of a function. Find the intervals where the *function* is increasing/decreasing. Determine the critical points and whether there are extreme values at them.

3. Find the intervals where $f(x)$ is increasing/decreasing and the relative minimum and maximum values (if any).

   (a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3$
   (b) $f(x) = 6\sqrt[3]{(x-2)^2}$
   (c) $f(x) = \frac{2x}{x^2+4}$
   (d) $f(x) = e^x(x^2 - x - 5)$

4. Consider the object whose position (in meters) is a function of time (in seconds) given by the formula $s(t) = t^3 - 7t^2 + 13t$. Determine the times the object changes the direction of the motion and how far from the starting point the object is at those times.

5. The percent concentration of a certain medication during the first 20 hours after it has been administered is approximated by

$$p(t) = \frac{230t}{t^2 + 6t + 9} \qquad 0 \le t \le 20$$

Determine at which hour is the concentration maximal and what the maximal percent concentration is.

6. A company determines that its revenue function is $R(x) = 15.22xe^{-.015x}$. Determine the production level which produces the maximal revenue and find that maximal revenue.

Solutions.

1. (a) The function is increasing on (-1, 1) and decreasing on $(-\infty, -1)$ and $(1, \infty)$. The critical points are $x = 1$ and $x = -1$ and the function has extreme values at ± 1. At 1, f has a maximum value $f(1) = 2$ and at -1 f has a minimum value $f(-1) = -2$.

   (b) The function is increasing on $(-\infty, -3)$ and $(-3, 2)$ and decreasing on $(2, \infty)$. The critical points are $x = -3$ and $x = 2$ but since the function is not define at them, there are no extreme values.

   (c) The function is increasing on $(-\infty, -2)$, $(-2, -1.2)$ and $(0, 1)$. The function is decreasing on $(-1.2, 0)$ and $(1, \infty)$. The critical points are $-2, -1.2, 0$ and 1. At -1.2 and 1, there are maximum values of $f(-1.2) = 2.8$ and $f(1) = 1$. At -2 and 0 there are no extreme values: at -2 the derivative does not change the sign and at 0 the function is not continuous because $\lim_{x\to 0} = -2$ and $f(0) = 2$. So the function never reaches the lowest value of -2 which it is approaching when $x \to 0$.

   (d) The function is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$ and $(-1, 1)$. The critical points are -1 and 1. At 1, there is a minimum values of $f(1) = 0$. At -1 there is no extreme value since the derivative does not change the sign.

2. For increasing/decreasing intervals, look for the intervals on which the derivative is positive/negative. (a) The derivative on the first graph is positive on $(-\infty, -2)$ and $(0, 2.5)$ so the function is increasing then. The derivative is negative on $(-2, 0)$ and $(2.5, \infty)$ so the function is decreasing then. $-2, 0$ and 2.5 are critical points. At -2 and 2.5 the function is changing from increasing to decreasing so at these points there are maximum values. At 0 the function is changing from decreasing to increasing so there is a minimum value at 0.

   (b) The derivative on the first graph is positive on $(-5, -1)$ and $(2, \infty)$ so the function is increasing then. The derivative is negative on $(-\infty, -5)$ and $(-1, 2)$ so the function is decreasing then. $-5, -1$ and 2 are critical points. At -1 the function is changing from increasing to decreasing so there is a maximum value at -1. At -5 and 2 the function is changing from decreasing to increasing so there are minimum values at those points.

3. (a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3 \Rightarrow f'(x) = x^2 + 2x - 15 = (x+5)(x-3)$. Setting the derivative to zero gives you the critical points $x = -5$ and $x = 3$. The number line test for f' produces $\begin{array}{ccccc} + & & - & & + \\ \hline & \text{-5} & & 3 & \end{array}$. Hence, f is increasing on $(-\infty, -5)$ and $(3, \infty)$, and decreasing on $(-5, 3)$. There is a relative minimum of -24 at $x = 3$ and a relative maximum of $\frac{184}{3}$ ≈61.33 at $x = -5$.

   (b) $f(x) = 6\sqrt[3]{(x-2)^2} \Rightarrow f'(x) = 6\frac{2}{3}(x-2)^{-1/3} = \frac{4}{\sqrt[3]{x-2}}$. f' is not defined at 2 and it is never zero so 2 is the only critical point. The derivative is changing the sign at 2 from negative to positive. Thus there is a minimum of $f(2) = 0$ at $x = 2$. f is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

(c) $f(x) = \frac{2x}{x^2+4} \Rightarrow f'(x) = \frac{2(x^2+4)-2x(2x)}{(x^2+4)^2} = \frac{8-2x^2}{(x^2+4)^2} = \frac{2(2-x)(2+x)}{(x^2+4)^2}$. Since the denominator is never zero, the only critical points are 2 and -2. The number line test gives you $\underset{\quad -2 \qquad 2}{\overline{-\qquad + \qquad -}}$ and so f is increasing on $(-2, 2)$ and decreasing on $(-\infty, 2)$ and $(2, \infty)$. There is a maximum of $\frac{1}{2}$ at $x = 2$ and a minimum of $\frac{-1}{2}$ at $x = -2$.

(d) $f(x) = e^x(x^2 - x - 5) \Rightarrow f'(x) = e^x(x^2 - x - 5) + e^x(2x - 1) = e^x(x^2 - x - 5 + 2x - 1) = e^x(x^2 + x - 6) = e^x(x + 3)(x - 2)$. Since e^x is never zero, 2 and -3 are the only critical points. From the number line test $\underset{\quad -3 \qquad 2}{\overline{+\qquad - \qquad +}}$, we have that f is increasing on $(-\infty, -3)$ and $(2, \infty)$ and decreasing on $(-3, 2)$. At $x = -3$ there is a maximum of $f(-3) = 7e^{-3} \approx .348$ and at $x = 2$ there is a minimum of $f(2) = -3e^2 \approx -22.17$.

4. The direction of the motion changes when the position function is changing from increasing to decreasing or vice versa. So, those times corresponds to critical points at which there are extreme values. $s(t) = t^3 - 7t^2 + 13t \Rightarrow v(t) = s'(t) = 3t^2 - 14t + 13$. Use the calculator program or the quadratic equation formula to determine that $3t^2 - 14t + 13 = 0 \Rightarrow t = \frac{14 \pm \sqrt{40}}{6} = \frac{7 \pm \sqrt{10}}{3} \Rightarrow t = 3.39$ and $t \approx 1.30$. Perform the number line test and obtain that the motion changes the direction from forwards to backwards at 1.30 seconds and from backwards to forwards at 3.39 seconds after it left the initial position. When $t = 1.3$ seconds, the object is about 7.27 meters from the starting point and when $t = 3.39$ seconds, the object is 2.58 meters from the starting point. Thus the distance increases from 0 to 7.27 meters in the first 1.3 seconds, then decreases from 7.27 to 2.58 meters from 1.3 to 3.39 seconds and then increases again after 3.39 seconds.

5. $p(t) = \frac{230t}{t^2+6t+9} \Rightarrow p'(t) = \frac{230(t^2+6t+9)-(2t+6)230t}{(t^2+6t+9)^2} = \frac{230(t^2+6t+9-2t^2-6t)}{(t^2+6t+9)^2} = \frac{230(9-t^2)}{(t^2+6t+9)^2} = \frac{230(3-t)(3+t)}{(t+3)^4} = \frac{230(3-t)}{(t+3)^3}$. Thus the critical points are ± 3. Perform the number line test $\underset{\quad -3 \qquad 3}{\overline{-\qquad + \qquad -}}$. Determine that there is a maximum at 3. The function f is not defined at -3, so there are no extreme value at -3 (the negative time values are not relevant in the context of the problem anyway). Thus the concentration is maximal 3 hours after the medication is administered and the maximal percent concentration is $p(3) = \frac{115}{6} \approx 19.17$ %.

6. $R(x) = 15.22xe^{-.015x} \Rightarrow R'(x) = 15.22e^{-.015x} + 15.22xe^{-.015x}(-.015) = 15.22e^{-.015x}(1 - .015x)$. Since the term in front of parenthesis is never zero, the only critical point is $1 - .015x = 0 \Rightarrow x = \frac{200}{3} \approx 66.67$. Perform the number line test to show that the function is changing from increasing to decreasing at this point. Thus, there is a maximal value at this point. The maximum value is $R(66.67) \approx 373.27$. When interpreting this answer you can round the integer x-value assuming that the entire items are produced. So, the maximal revenue of \$373.27 is obtained when 67 items are sold.