

Antiderivative and The Indefinite Integral

The idea of employing an inverse function is central when solving any equation:

- $x + 2 = 4$ requires application of the function $x - 2$, the inverse of $x + 2$, to the right hand side. Thus $x = 4 - 2 = 2$.
- $x^2 = 4$ requires application of the function \sqrt{x} (and $-\sqrt{x}$ for the left branch), the inverse of x^2 , to the right hand side. Thus $x = \pm\sqrt{4} = \pm 2$.
- $2^x = 4$ requires application of the function $\log_2 x$, the inverse of 2^x , to the right hand side. Thus $x = \log_2 4 = 2$.

The same argument can be employed when looking for an unknown *function* such that its derivative is a given function.

Given a function $f(x)$, find function y such that $y' = f(x)$.

Finding a solution of this general problem requires the use of the *inverse of derivation*, that is the *process that produces a function whose derivative is known*. This process is known as the **antidifferentiation** and the outcome of this process is known as an **antiderivative**. Thus,

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$.

The antidifferentiation is relevant when the rate of change of a quantity can be measured and the quantity size itself needs to be determined from the rate. For example, if velocity is known and we need to determine the function computing the distance traveled, the antiderivation is needed.

In some cases, an antiderivative can be easily guessed.

Example 1. Find a function with derivative 4.

Solution. To find the solution, one needs to note that any line with slope 4 has derivative 4. Thus, any line of the form $y = 4x + c$ is the solution of this problem.

Example 2. Find a function with derivative x^2 .

Solution. Since a cubic function has a quadratic function for derivative, that is our prime suspect. An adjustment is needed since derivative of x^3 is $3x^2$ not x^2 . Dividing the cubic with 3 fixes this: $\frac{1}{3}x^3$ indeed has derivative $\frac{1}{3}3x^2 = x^2$.

Note that $\frac{1}{3}x^3 + c$ with c any constant also has derivative x^2 .

The previous examples indicate that if $F(x)$ is an antiderivative of $f(x)$ then so is $F(x) + c$. Indeed if $\frac{d}{dx}F(x) = f(x)$ then $\frac{d}{dx}(F(x) + c) = f(x)$ as well since the derivative of a constant is zero.

Conversely, we claim that any antiderivative of $f(x)$ has to have the form $F(x) + c$ for some constant c . Indeed, if $G(x)$ is another antiderivative of $f(x)$, then $\frac{d}{dx}(G(x) - F(x)) = f(x) - f(x) = 0$. Since only a constant function has slope of the tangent equal to 0 at every point (this intuitive fact follows from the Mean Value Theorem), we have that $G(x) - F(x) = c$ for some constant c and so $G(x) = F(x) + c$.

The collection of *all* antiderivatives of a function $f(x)$ is called the **indefinite integral** and it is denoted by

$$\int f(x) dx$$

Thus, we have the following.

$$\text{If } F(x)' = f(x) \text{ then } \int f(x)dx = F(x) + c.$$

The process of producing the indefinite integral for a given function $f(x)$ is known as **integration** and the function $f(x)$ is known as the **integrand**.

The definition above implies that to integrate a function, one needs to find any antiderivative of the function and adds a constant to it. The definition above also gives you that you can **cancel** the derivative and integral in the following way.

$$\int \left(\frac{d}{dx} F(x) \right) dx = F(x) + c \quad \text{and} \quad \frac{d}{dx} \left(\int f(x) dx \right) = f(x).$$

While guessing an antiderivative of a function may work for some simple functions, this method is not always effective. Because of that we develop several rules which are used to simplify the process of finding antiderivative.

The first rule we present follows the idea of Example 2. To find an antiderivative of x^n , we divide the function x^{n+1} with $n + 1$. Then indeed derivative of $\frac{1}{n+1}x^{n+1}$ is $\frac{1}{n+1}(n + 1)x^n = x^n$ as needed. Thus $\frac{1}{n+1}x^{n+1} + c$ is an antiderivative of x^n for any constant c .

$$\text{The Power Rule.} \quad \int x^n dx = \frac{1}{n+1}x^{n+1} + c$$

This rule can be applied to any real value of n except $n = -1$. We shall return to the case $n = -1$ later.

The second two rules follow from sum and constant multiple rules of differentiation. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ an antiderivative of $g(x)$, then $(F(x) + G(x))' = F'(x) + G'(x) = f(x) + g(x)$ so that $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$. This gives us the following.

$$\text{The Sum Rule.} \quad \int (f(x) + g(x))dx = \int f(x) dx + \int g(x)dx$$

Using similar arguments, one shows that if k is a constant and $F(x)$ is an antiderivative of $f(x)$, then $kF(x)$ is an antiderivative of $kf(x)$ so that the following holds.

$$\text{The Constant Multiple Rule.} \quad \int kf(x)dx = k \int f(x) dx$$

Note that the sum and the constant multiple rule also imply that $\int (f(x) - g(x))dx = \int f(x) dx + \int -g(x)dx = \int f(x) dx + (-1) \int g(x)dx = \int f(x) dx - \int g(x)dx$.

These two rules enables us to find antiderivative of any polynomial function by **integrating term by term**. It is important to keep the following in mind.

1. Keep carrying *both* the integral sign \int and the symbol dx until you use an integration rule (just the power rule for the time being) which eliminates \int and dx *simultaneously*.
2. When \int and dx are eliminated, *do not forget to add a constant* to your answer.

The next example illustrates integration of polynomials.

Example 3. Find the integral $\int (6x^2 - 4x + 5)dx$.

Solution. Using the sum and constant multiple rule, we have that

$$\int (6x^2 - 4x + 5)dx = \int 6x^2 dx - \int 4x dx + \int 5 dx = 6 \int x^2 dx - 4 \int x dx + 5 \int dx$$

The first two integrals can be evaluated using the power rule. The last integral can be evaluated either using the power rule and representing the integrand 1 as x^0 and using the power rule (producing $\frac{1}{0+1}x^{0+1} = 1x^1 = x$) or simply noting that x is an antiderivative of 1 since derivative of 1 is x . In either case, we obtain

$$6 \int x^2 dx - 4 \int x dx + 5 \int dx = 6 \frac{1}{2+1} x^{2+1} - 4 \frac{1}{1+1} x^{1+1} + 5x + c = 2x^3 - 2x^2 + 5x + c.$$

Keep in mind the following algebra rules can be handy when integrating non-polynomial power functions.

$\frac{1}{x^n} = x^{-n}$	$\sqrt[n]{x} = x^{1/n}$
--------------------------	-------------------------

Example 4. Find the integral $\int \left(\frac{\sqrt{x^3}}{3} + \frac{3}{\sqrt[3]{x}} \right) dx$.

Solution. Write the function $\frac{\sqrt{x^3}}{3} + \frac{3}{\sqrt[3]{x}}$ as $\frac{1}{3}x^{3/2} + 3x^{-1/3}$ so that the power rule applies to each term. Integrate each term and obtain

$$\frac{1}{3} \int x^{3/2} dx + 3 \int x^{-1/3} dx = \frac{1}{3} \frac{1}{\frac{3}{2}+1} x^{3/2+1} + 3 \frac{1}{\frac{-1}{3}+1} x^{-1/3+1} + c = \frac{2}{15} x^{5/2} - \frac{9}{2} x^{2/3} + c.$$

You can also express your answer as $\frac{2}{15}\sqrt{x^5} - \frac{9}{2}\sqrt[3]{x^2} + c$.

The next example illustrates that in some cases you need to simplify the function before you can apply the above rules.

Example 5. Find the integral $\int \frac{3x^4+7}{x^2} dx$.

Solution. Write the function as $\frac{3x^4}{x^2} + \frac{7}{x^2} = 3x^2 + 7x^{-2}$ so that the power rule applies to each term. Integrate each term and obtain

$$3 \int x^2 dx + 7 \int x^{-2} dx = 3 \frac{1}{2+1} x^{2+1} + 7 \frac{1}{-2+1} x^{-2+1} + c = x^3 - 7x^{-1} + c = x^3 - \frac{7}{x}.$$

In some cases, we need to find the particular antiderivative satisfying particular condition instead of all the antiderivatives. The next example illustrates this.

Example 6. Find the function $f(x)$ with derivative $x^2 - \sqrt{x}$ such that $f(1) = 3$.

Solution. We can find $f(x)$ as the antiderivative of $x^2 - \sqrt{x}$ which passes (1,3). Thus, we can

1. Find $f(x)$ as the indefinite integral of the derivative $x^2 - \sqrt{x}$;
2. Then find the value of c such that $f(1) = 3$.

Step 1. gives us that $f(x) = \int (x^2 - \sqrt{x}) dx = \frac{1}{2+1}x^{2+1} - \frac{1}{\frac{1}{2}+1}x^{1/2+1} + c = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + c$. To do step 2, note that since $f(1) = 3$, $3 = \frac{1}{3}1^3 - \frac{2}{3}1^{3/2} + c = \frac{1}{3} - \frac{2}{3} + c \Rightarrow 3 = \frac{-1}{3} + c \Rightarrow 10 = 3c \Rightarrow c = \frac{10}{3}$. Thus

$$f(x) = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + \frac{10}{3}.$$

Applications.

Finding an antiderivative applies to every situation when the rate is given and the function with the given rate is needed. For example, recall that the velocity is the derivative of the distance function thus, the *distance is an antiderivative of the velocity*. Similarly, since the acceleration is the derivative of velocity, the *velocity is an antiderivative of the acceleration*.

Example 7. An object is dropped from a cliff 300 meters high. Determine the time the object hits the ground and find the speed at the time of the impact. You can neglect the drag force and assume that the gravitational force is the only force that acts on an object producing the acceleration of 9.8 m/s^2 .

Solution. If we consider $s(t)$ to be the distance from the ground to the object, then $s(t)$ is decreasing so that the velocity $v(t) = s'(t)$ is negative. Since the speed is increasing and $v(t)$ is negative, $v(t)$ is becoming more negative as time passes by so $v(t)$ is decreasing. Thus, $a(t) = v'(t)$ is negative. Hence

$$a(t) = v'(t) = -9.8 \Rightarrow v(t) = \int -9.8 dt = -9.8t + c$$

Since the object is dropped, the initial velocity is zero and so we need the antiderivative such that $v(0) = 0$. Hence $c = 0$ and

$$v(t) = s'(t) = -9.8t \Rightarrow s(t) = \int -9.8t dt = -9.8 \frac{1}{2}t^2 + c = -4.9t^2 + c$$

Since $s(0) = 300$, we have that $300 = -4.9(0) + c \Rightarrow c = 300$. Thus

$$s(t) = -4.9t^2 + 300.$$

The object hits the ground when $s(t) = 0 \Rightarrow 300 = 4.9t^2 \Rightarrow t^2 = \frac{300}{4.9} \Rightarrow t \approx 7.82$ seconds. The velocity at the time of the impact is $v(7.82) = -76.68 \text{ m/s}$ so the speed is 76.68 meters per second.

Practice Problems.

1. Evaluate the following integrals.

$$(a) \int (x^5 + 2) dx \quad (b) \int (4x^3 + 6x^2 - 4x + 3) dx \quad (c) \int (\sqrt{x} - \frac{4}{x^2}) dx$$

$$(d) \int \frac{1+x^3}{x^2} dx \qquad (e) \int \frac{8\sqrt[3]{x+x^2}}{4} dx \qquad (f) \int \frac{x+3}{\sqrt{x}} dx$$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

(a) $f'(x) = \frac{1}{\sqrt[3]{x}}$ and $f(8) = 9$

(b) $f'(x) = 5\sqrt{x^3} + 3$ and $f(1) = 4$

3. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{2}$ where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 2$ seconds, the position function $s(t) = 5$ feet, determine the position function $s(t)$.

4. A stone is being thrown up in the air with initial velocity of 5 m/s. Determine the time the object hits the ground and find the speed at the time of the impact. Determine also the maximal height it reaches. You can neglect the drag force and assume that the gravitational force is the only force that acts on an object producing the constant acceleration of 9.8 m/s².

Solutions.

1. (a) $\int (x^5 + 2) dx = \int x^5 dx + \int 2 dx = \frac{1}{6}x^6 + 2x + c.$

(b) $\int (4x^3 + 6x^2 - 4x + 3) dx = 4\frac{1}{4}x^4 + 6\frac{1}{3}x^3 - 4\frac{1}{2}x^2 + 3x + c = x^4 + 2x^3 - 2x^2 + 3x + c.$

(c) $\int (x^{1/2} - 4x^{-2}) dx = \frac{2}{3}x^{3/2} + \frac{4}{x} + c$

(d) $\int \frac{1+x^3}{x^2} dx = \int (\frac{1}{x^2} + \frac{x^3}{x^2}) dx = \int (x^{-2} + x) dx = -\frac{1}{x} + \frac{x^2}{2} + c$

(e) $\int \frac{8\sqrt[3]{x+x^2}}{4} dx = \int (2x^{1/3} + \frac{1}{4}x^2) dx = 2\frac{3}{4}x^{4/3} + \frac{1}{4} \cdot \frac{1}{3}x^3 + c = \frac{3}{2}\sqrt[3]{x^4} + \frac{1}{12}x^3 + c$

(f) $\int \frac{x+3}{\sqrt{x}} dx = \int (\frac{x}{\sqrt{x}} + \frac{3}{\sqrt{x}}) dx = \int x^{1/2} dx + \int 3x^{-1/2} dx = \frac{1}{3/2}x^{3/2} + 3\frac{1}{1/2}x^{1/2} = \frac{2}{3}\sqrt{x^3} + 6\sqrt{x} + c$

2. (a) $f(x) = \int f'(x) dx = \int x^{-1/3} dx = \frac{3}{2}\sqrt[3]{x^2} + c.$ Using $f(8) = 9$ to solve for c , you have that $9 = \frac{3}{2}\sqrt[3]{8^2} + c = \frac{3}{2} \cdot 4 + c = 6 + c \Rightarrow 9 = 6 + c \Rightarrow c = 3.$ Thus, $f(x) = \frac{3}{2}\sqrt[3]{x^2} + 3.$

(b) $f(x) = \int f'(x) dx = \int (5x^{3/2} + 3) dx = 5\frac{2}{5}x^{5/2} + 3x + c = 2\sqrt{x^5} + 3x + c.$ Using $f(1) = 4$ to solve for c , you have that $4 = 2\sqrt{1^5} + 3(1) + c = 2 + 3 + c = 5 + c \Rightarrow 4 = 5 + c \Rightarrow c = -1.$ Thus, $f(x) = 2\sqrt{x^5} + 3x - 1.$

3. Recall that $s(t) = \int v(t) dt.$ Thus $s(t) = \frac{1}{2} \frac{t^2}{2} + c = \frac{t^2}{4} + c.$ Using that $s(2) = 5$, we have that $5 = \frac{2^2}{4} + c = 1 + c \Rightarrow c = 4.$ Thus $s(t) = \frac{t^2}{4} + 4.$

4. The velocity is decreasing and positive during the time the object goes up. When the object starts falling, the velocity is negative and becoming more and more negative, that is, it is also decreasing. Thus the acceleration is negative throughout the motion. $a(t) = v'(t) = -9.8 \Rightarrow v(t) = \int -9.8 dt = -9.8t + c.$ Since the initial velocity is 5, $v(0) = 5 \Rightarrow 5 = -9.8(0) + c \Rightarrow c = 5$ and so $v(t) = s'(t) = -9.8t + 5.$ Obtain the distance function as $s(t) = \int (-9.8t + 5) dt = -4.9t^2 + 5t + c.$ Since $s(0) = 0$, we have that $c = 0$ and so $s(t) = -4.9t^2 + 5t.$

The object hits the ground when $s(t) = 0$ and $t > 0.$ Since $s(t)$ factors as $t(-4.9t + 5)$ the nonzero solution is $5 = 4.9t \Rightarrow t = 1.02$ seconds. The velocity at that time is $v(1.02) = -5$ m/s so the final speed is the same as the initial speed.

The maximal height can be obtained as the value of s at the critical point (i.e. time when the velocity is zero). $v(t) = -9.8t + 5 = 0 \Rightarrow t = \frac{5}{9.8} = 0.51$ second. $s(\frac{5}{9.8}) \approx 1.28$ meters.