Calculus 1
Lia Vas

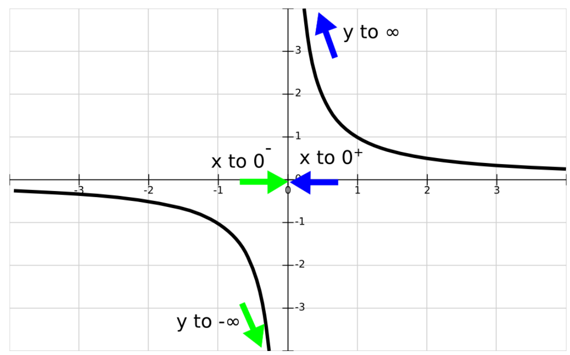
Infinite Limits and Limits at Infinity

Horizontal and vertical asymptotes

Example 1. Consider the function $f(x) = \frac{1}{x}$. This function is defined for every value of x except $x = 0$. Although not defined at 0, it still may be relevant to know the *behavior* of this function *near* zero. Thus, we consider $\lim_{x\to 0} f(x)$.

Solution. Note that when x is taking smaller and smaller positive values, the y-values increase without bound. Thus

$$\text{when } x \to 0^+,\ y \to \infty.$$



This conclusion is supported by consideration of the graph of $f(x)$ as well. Similarly, when x is taking negative values closer and closer to 0, the y-values become larger and larger negative values. Thus

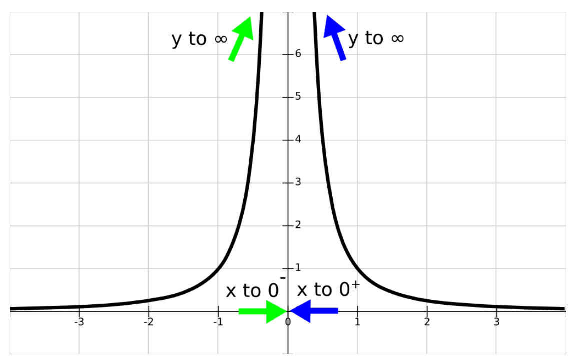
$$\text{when } x \to 0^-,\ y \to -\infty.$$

Since the left limit is $-\infty$ and the right limit is $+\infty$, the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Example 2. Consider the function $f(x) = \frac{1}{x^2}$ now, examine the behavior of this function at 0 and find $\lim_{x\to 0} f(x)$.

Solution. Note that when x is taking smaller and smaller values, both positive and negative, the y-values become larger and larger positive values. Thus

$$y \to \infty \text{ both when } x \to 0^+ \text{ and when } x \to 0^-.$$



Support this conclusion by considering the graph of $f(x)$ too.

Since left and right limits are equal, the limit $\lim_{x\to 0} \frac{1}{x^2}$ exists (bot not as a finite number) and it is equal to ∞.

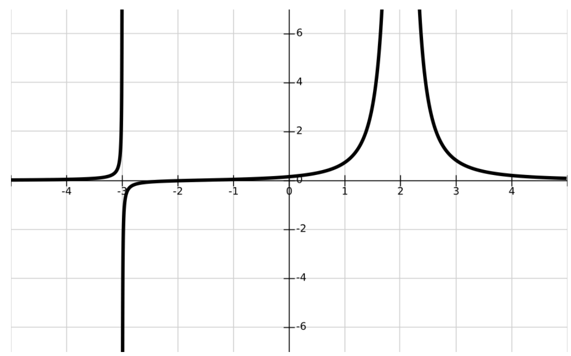
Infinite limits. Consider the first example again, when $x \to 0^+$, the function $\frac{1}{x}$ takes large positive values so the limit is ∞. We can write down this conclusion as $\frac{1}{0^+} = \infty$. Similarly, when $x \to 0^-$, $\frac{1}{x} \to -\infty$ which we can write as $\frac{1}{0^-} = -\infty$. From these observations, we can conclude the following.

If the numerator is nonzero, the expressions 0^- or 0^+ in denominator cause a limit not to have a finite value.
If $a > 0$, $\frac{a}{0^\pm} = \pm\infty$. If $a < 0$, $\frac{a}{0^\pm} = \mp\infty$

We illustrate the above observations by the following example.

Example 3. Determine $\lim_{x\to -3} f(x)$ and $\lim_{x\to 2} f(x)$ for $f(x) = \frac{x+2}{(x+3)(x-2)^2}$.

Solution. Let us find the left and right limits at -3 first. When $x \to -3^-$, $f(x) \to \frac{-1}{(-3^-+3)(-3-2)^2} = \frac{-1}{(0^-)(25)}$. The sign of the answer is positive since we are dividing a negative with a negative number. Thus, the answer is ∞.



When $x \to -3^+$, $f(x) \to \frac{-1}{(-3^++3)(-3-2)^2} = \frac{-1}{(0^+)(25)}$. The sign of the answer is negative since we are dividing a negative with a positive number. Thus, the answer is $-\infty$.

Consider the graph of $f(x)$ too to validate your conclusions.

In examples 1 and 2, the y-values are approaching the vertical line $x = 0$ (y-axis) for $x \to 0$. In this case, we refer $x = 0$ as the vertical asymptote. Thus,

The vertical line $x = a$ is a **vertical asymptote** of a function $f(x)$ if either $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ are equal to ∞ or $-\infty$.

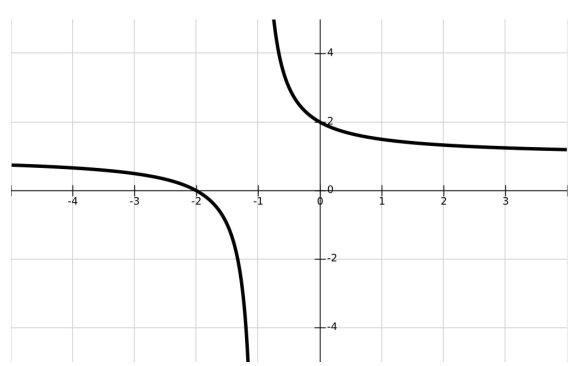
Note that the function from example 3 has two vertical asymptotes $x = -3$ and $x = 2$.

To find the vertical asymptotes of a function, consider the x-values at which the function is not defined as you and note that a function may not have a vertical asymptote at all such x-values.

Example 4. Find vertical asymptotes of $f(x) = \frac{x^2-x-6}{x^2-2x-3}$.

Solution. To determine the points at which $f(x)$ is not defined, factor the numerator and denominator and obtain $f(x) = \frac{(x-3)(x+2)}{(x-3)(x+1)}$. This tells you that there are two potential vertical asymptotes $x = 3$ and $x = -1$. Determine limits at both values.

Note that when $x \neq 3$, the terms $x - 3$ cancel and $f(x)$ is equal to $\frac{x+2}{x+1}$. So the limit $\lim_{x\to 3}$ is $\frac{3+2}{3+1} = \frac{5}{4}$. Since this is a finite value, there is no vertical asymptote at 3.



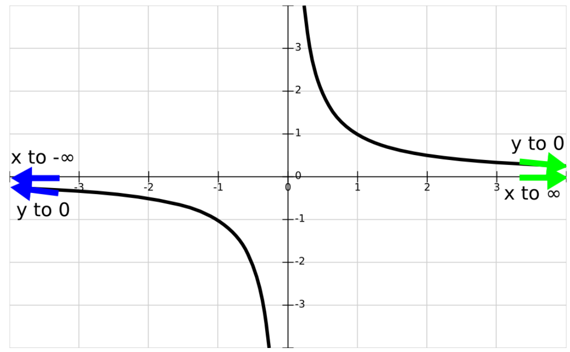
When $x \to -1^+$, $f(x) \to \frac{-1+2}{-1^++1} = \frac{1}{0^+} = \infty$. Since this is not a finite value, there is a vertical asymptote at -1. The left limit is also not finite, $\lim_{x\to -1^-} f(x) = \frac{1}{0^-} = -\infty$ indicating that $f(x)$ approaches the vertical asymptote $x = -1$ from the left too.

Consider the graph of $f(x)$ to support your conclusions.

Limits at infinity. When x-values become arbitrarily large, we say that x approaches infinity and write $x \to \infty$. The limits $\lim_{x\to\infty} f(x)$ are relevant for determining the **long term behavior** of $f(x)$.

When x-values become arbitrarily large negative value, we say that x approaches negative infinity and write $x \to -\infty$.

Example 1 revisited. Find limits $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.



Solution. Note that when x is a large positive number, $\frac{1}{x}$ is a small (positive) number. Thus $\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$. Similarly, when x is a large negative number, $\frac{1}{x}$ is a small (negative) number. Thus $\lim_{x\to-\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$ as well.

Consider the graph of $f(x)$ to support your conclusions.

Example 3 revisited. Find limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x)$ from example 3.

Solution. Considering the graph of function from example 3, you can conclude that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

Limits at infinity of polynomials and rational functions. While it is easy to use the graph to determine the limit at infinity of the function from example 3, it may note be as straightforward to do the same for the function from example 4. From the graph alone (see the graph of function in example 4), we can see that the limit of $f(x)$ is a finite number, but it is not as obvious what this number is as it was in example 3. For cases like this, we need the following analysis.

When x is a large number, the value of a polynomial function is impacted the most by the term with the highest power of x, known also as the **leading term**. For example, consider the values of each term of the polynomial $x^2 - 2x - 3$ for $x = 1000$. The value of x^2 is 1000000, the value of $-2x$ is -2000 while the last term -3 has the same value for any x. We can see that the value of the leading term x^2 impacts the total value of the polynomial the most.

This observation can help us simplify determination of limits at infinity of rational functions a great deal. Recall that a rational function is a quotient of two polynomial functions. For example, all functions from examples 1–4 are rational functions.

> If $p(x)$ and $q(x)$ are polynomials with leading terms $a_n x^n$ and $b_m x^m$, then $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ is equal to $\lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$.

The same reasoning can be used for limits when $x \to -\infty$.

Example 4 revisited. Find the limit of $f(x) = \frac{x^2-x-6}{x^2-2x-3}$ when $x \to \pm\infty$.

Solution. Using the reasoning above $\lim_{x\to\pm\infty} \frac{x^2-x-6}{x^2-2x-3} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = 1$.

In examples 1, 2, and 3, the y-values are approaching the horizontal line $y = 0$ (x-axis) for $x \to \infty$. In this case, we refer $y = 0$ as the horizontal asymptote.

The horizontal line $y = b$ is a **horizontal asymptote** of a function $f(x)$ if $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ (possibly both).

For example, the function in example 4 has horizontal asymptote $y = 1$.

Horizontal asymptotes of a rational function. Recall that for the rational function $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials with leading terms $a_n x^n$ and $b_m x^m$, we have $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$. Since this limit is

1. 0 if $n < m$,
2. $\frac{a_n}{b_m}$ if $m = n$, and
3. $\pm\infty$ if $n > m$,

the rational function $\frac{p(x)}{q(x)}$ has a horizontal asymptote if $n \le m$.

Example 5. Determine the horizontal asymptotes of the following the rational functions.

(a) $f(x) = \dfrac{57x^3 - 4x^2 + 9}{35x - 114x^3}$ (b) $f(x) = \dfrac{57x^5 - 4x^2 + 9}{35x - 114x^3}$ (c) $f(x) = \dfrac{57x^3 - 4x^2 + 9}{35x - 114x^4}$

Solution. (a) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{57x^3}{-114x^3} = \frac{57}{-114} = \frac{-1}{2}$ so $f(x)$ has the horizontal asymptote $y = \frac{-1}{2}$.

(b) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{57x^5}{-114x^3} = \lim_{x\to\pm\infty} \frac{57x^2}{-114} = -\infty$ so $f(x)$ does not have a horizontal asymptote.

(c) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{57x^3}{-114x^4} = \lim_{x\to\pm\infty} \frac{57}{-114x} = 0$ so $f(x)$ has the horizontal asymptote $y = 0$.

Practice problems.

1. Evaluate the following limits.

(a) $\lim_{x\to3^-} \frac{2}{x-3}$ (b) $\lim_{x\to3} \frac{2}{x-3}$ (c) $\lim_{x\to\infty} \frac{2}{x-3}$
(d) $\lim_{x\to\infty} \frac{2x}{x-3}$ (e) $\lim_{x\to\infty} \frac{x-1}{x^2-3x+2}$ (f) $\lim_{x\to\infty} \frac{x^2-x-6}{x^2-2x-3}$
(g) $\lim_{x\to0^+} \frac{x+2}{x(x-2)}$ (h) $\lim_{x\to2} \frac{x+2}{x(x-2)}$ (i) $\lim_{x\to\infty} \frac{x+2}{x(x-2)}$

2. Find the horizontal and vertical asymptotes (if any) of the following functions.

(a) $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - x - 6}$ (b) $f(x) = \dfrac{x + 5}{x^2 - x - 6}$ (c) $f(x) = \dfrac{x^3 + 1}{x(3 - x)}$

Solutions.

1. Graph all the functions and consider the graphs.

(a) $\lim_{x\to3^-} \frac{2}{x-3} = \frac{2}{0^-} = -\infty$

(b) By part (a), the left limit is $-\infty$. The right limit is $\lim_{x\to3^+} \frac{2}{x-3} = \frac{2}{0^+} = \infty$. Since left and right limits are different, the limit $x \to 3$ does not exist.

(c) $\lim_{x\to\infty} \frac{2}{x-3} = \frac{2}{\infty} = 0$.

(d) $\lim_{x\to\infty} \frac{2x}{x-3} = \lim_{x\to\infty} \frac{2x}{x} = 2$.

(e) $\lim_{x\to\infty} \frac{x-1}{x^2-3x+2} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$.

(f) $\lim_{x\to\infty} \frac{x^2-x-6}{x^2-2x-3} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1$.

(g) $\lim_{x\to 0^+} \frac{x+2}{x(x-2)} = \frac{2}{0^+(-2)} = \frac{1}{-0^+} = -\infty$.

(h) Compare the left and and right limits $\lim_{x\to 2^+} \frac{x+2}{x(x-2)} = \frac{4}{2(0^+)} = \infty$ and $\lim_{x\to 2^-} \frac{x+2}{x(x-2)} = \frac{4}{2(0^-)} = -\infty$. Thus $\lim_{x\to 2} \frac{x+2}{x(x-2)}$ does not exist.

(i) $\lim_{x\to\infty} \frac{x+2}{x(x-2)} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$.

2. (a) To find the vertical asymptotes, simplify $f(x)$ as $\frac{x^2-2x-8}{x^2-x-6} = \frac{(x-4)(x+2)}{(x-3)(x+2)}$. Thus, $f(x)$ is not defined when $x = 3$ and $x = -2$. When $x \neq -2$, $f(x) = \frac{x-4}{x-3}$ and the limit $\lim_{x\to -2} f(x) = \frac{-2-4}{-2-3} = \frac{6}{5}$. Since this is a finite value, $f(x)$ does not have a vertical asymptote at $x = -2$. $\lim_{x\to 3^+} f(x) = \frac{3-4}{0^+} = -\infty$ and $\lim_{x\to 3^-} f(x) = \frac{3-4}{0^-} = \infty$. Since these values are not finite, the line $x = 3$ is a vertical asymptote of $f(x)$.

   To find the horizontal asymptote, consider $\lim_{x\to\pm\infty} \frac{x^2-2x-8}{x^2-x-6} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = 1$. Thus $y = 1$ is a horizontal asymptote.

   (b) To find the vertical asymptotes, simplify $f(x)$ as $= \frac{x+5}{x^2-x-6} = \frac{x+5}{(x-3)(x+2)}$. Thus, $f(x)$ is not defined when $x = 3$ and $x = -2$. $\lim_{x\to -2^+} f(x) = \frac{-2+5}{(-2-3)(0^+)} = \frac{3}{(-5)0^+} = -\infty$ and $\lim_{x\to -2^-} f(x) = \frac{3}{(-5)0^-} = \infty$. Since these values are not finite, the line $x = -2$ is a vertical asymptote of $f(x)$. $\lim_{x\to 3^+} f(x) = \frac{3+5}{(0^+)(3+2)} = \frac{8}{0^+(5)} = \infty$ and $\lim_{x\to 3^-} f(x) = \frac{8}{0^-(5)} = -\infty$. Since these values are not finite, the line $x = 3$ is a vertical asymptote of $f(x)$.

   To find the horizontal asymptote, consider $\lim_{x\to\pm\infty} \frac{x+5}{x^2-x-6} = \lim_{x\to\pm\infty} \frac{x}{x^2} = 0$. Thus $y = 0$ is a horizontal asymptote.

   (c) $f(x)$ is not defined when $x = 3$ and $x = 0$. $\lim_{x\to 3^+} f(x) = \frac{27+1}{(3)(3-3^+)} = \frac{28}{(3)0^-} = -\infty$ and $\lim_{x\to 3^-} f(x) = \frac{28}{(3)0^+} = \infty$. Since these values are not finite, the line $x = 3$ is a vertical asymptote of $f(x)$. $\lim_{x\to 0^+} f(x) = \frac{0+1}{(0^+)(3-0)} = \frac{1}{0^+(3)} = \infty$ and $\lim_{x\to 0^-} f(x) = \frac{1}{0^-(3)} = -\infty$. Since these values are not finite, the line $x = 0$ is a vertical asymptote of $f(x)$.

   To find the horizontal asymptote, consider $\lim_{x\to\pm\infty} \frac{x^3+1}{x(3-x)} = \lim_{x\to\pm\infty} \frac{x^3}{-x^2} = -\infty$. Since this value is not finite $f(x)$ does not have a horizontal asymptote.