Calculus 1
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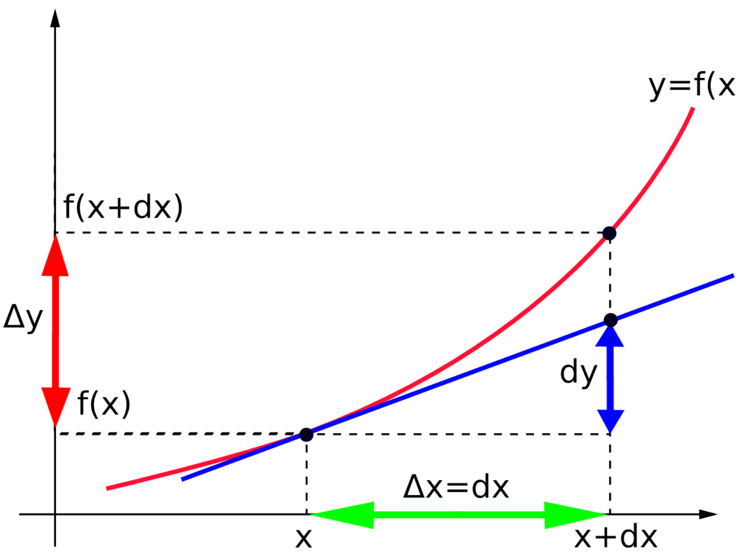
Linear Approximation

The differential. Consider a function $y = f(x)$ and the two points $(x, f(x))$ and $(x+h, f(x+h))$ on its graph. Recall that Δx is sometimes used to denote the difference h between the x-values and Δy is used for the difference $f(x+h) - f(x)$ of the y-values. In this notation, the derivative can be calculated as follows.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) \;\Rightarrow\; dy = f'(x)dx$$

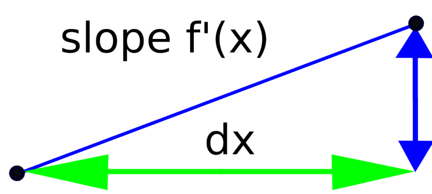
In this last formula, the quantity dy measures the rise (or decline) of the tangent line when x-values change by $dx = \Delta x$. The quantity $dy = f'(x)dx$ is known as the **differential** of the function $y = f(x)$.



Consider the geometric meaning of the change in y-values $\Delta y = f(x+h) - f(x)$ and the differential $dy = f'(x)dx$ on the figure on the right.

$$\boxed{dy = f'(x)dx}$$

This also agrees with the fact that the quotient $\frac{dy}{dx}$ is the tangent of the angle adjacent to dx which is the is the slope of the hypotenuse in the triangle on the right. This exactly corresponds to the derivative $\frac{dy}{dx} = f'(x)$.



Linear Approximation.

Although different in general, the change in y-values $\Delta y = f(x+h) - f(x)$ and the differential $dy = f'(x)dx$ are similar in size when dx is small. In this case, Δy and dy are approximately equal and we write

$$\Delta y \approx dy \Rightarrow f(x+h) - f(x) \approx f'(x)h \Rightarrow f(x+h) \approx f(x) + f'(x)h.$$

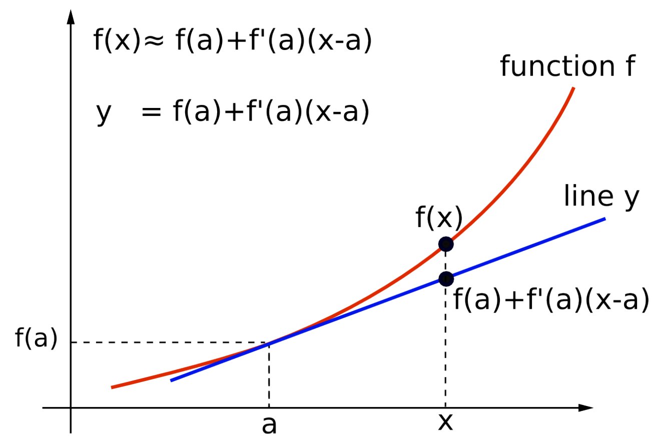
The last formula on the right represents the y-value of the tangent line at point $x+h$. This formula represents the point-slope equation if we denote $h+x$ by x and x by a, in which case $dx = x - a$, and consider Δy as $f(x) - f(a)$. The equation of line passing $(a, f(a))$ tangent to the graph of $f(x)$ has the slope $f'(a)$ so its point-slope equation is

$$y - f(a) = f'(a)(x-a) \Rightarrow y = f(a) + f'(a)(x-a).$$

The expression $f(a)+f'(a)(x-a)$ is called the **linear approximation** of $f(x)$ at $x=a$. Thus, approximating Δy by the differential dy amounts to approximating $f(x)$ by its linear approximation $f(a)+f'(a)(x-a)$.

$$\Delta y \approx dy \Leftrightarrow f(x)-f(a) \approx f'(a)(x-a) \Leftrightarrow$$

$$\boxed{f(x) \approx f(a)+f'(a)(x-a)}$$



In applications, you can think of the value $f(a)$ as of the **present value**, the value $f(x)$ then represents the **future value**, $(x-a)$ the **time lapsed** and $f'(a)$ the **change rate**. Thus

$f(x)$	$\approx$	$f(a)$	$+$	$f'(a)$	$(x-a)$
future value	$\approx$	present value	$+$	change rate	time elapsed

The linear approximation is particularly useful when certain phenomena is modeled by a function which is either

- too complex to be manipulated or
- such that its exact formulas is not known.

but its value and the value of its derivative are known at a point. We illustrate both scenarios in the next two examples.

Example 1. Find the linear approximation of e^x at 0 and use it to approximate $e^{0.01}$ by a rational number.

Solution. Let $f(x)=e^x$ so that $f'(x)=e^x$. Thus $f(0)=e^0=1$ and $f'(0)=e^0=1$. The linear approximation $f(x)\approx f(a)+f'(a)(x-a)$ for $f(x)=e^x$ and $a=0$ gives you that

$$e^x \approx 1+1(x-0)=1+x.$$

Note that this answer also indicates that the line $1+x$ is tangent to the graph of e^x at $x=0$.

When $x=0.01$ this approximation gives you $e^{0.01}\approx 1+0.01=1.01=\frac{101}{100}$. Comparing with the calculator value $e^{0.01}\approx 1.01005\ldots$ we can see that our approximation is accurate to first four decimal places.

Example 2. Approximate $f(3.06)$ and $f(2.9)$ given that $f(3)=1$ and $f'(3)=0.5$.

Solution. The linear approximation formula for $f(x)$ at 3 gives us that $f(x)\approx f(3)+f'(3)(x-3)$. Thus

$f(3.06)\approx f(3)+f'(3)(3.01-3)=1+0.5(0.06)=1.03$ and

$f(2.9)\approx f(3)+f'(3)(2.9-3)=1+0.5(-0.1)=0.95$.

Let us also consider a more applied example.

Example 3. The profit P of a company depends on the number of items x produced. The production level x depends on the time t (measured in years). Assume that the profit increases by \$200 with each new item produced and that the production level increases by 150 items each year.

(a) Determine the rate of increase of the profit per year.

(b) If the company is presently making a profit of \$800,000, approximate the profit in four year time.

Solution. (a) $\frac{dP}{dx} = 200$ dollars per item and $\frac{dx}{dt} = 150$ items per year. By the chain rule, $\frac{dP}{dt} = \frac{dP}{dx}\frac{dx}{dt} = 200 \cdot 150 = 30,000$ dollars per year.

(b) Using the linear approximation, $P(4) \approx P(0) + P'(0)(4-0) = 800,000 + 30,000(4) = 920,000$ dollars.

Practice problems.

1. Find the linear approximation of the following functions at the given x-value and use them to approximate given y-values.

   (a) $f(x) = \sin x$, $x = 0$, approximate $\sin 0.1$.

   (b) $f(x) = \ln(1 + 3x)$, $x = 0$, approximate $\ln 0.7 = \ln(1 - 3(0.1))$.

2. If $f(2) = 5$ and $f'(2) = 3$, approximate $f(2.1)$ and $f(1.9)$.

3. If $f(1) = 1$ and $f'(1) = -2$, approximate $f(1.01)$.

4. Use the linear approximation to estimate $\sqrt[3]{26}$ with a rational number. Compare to the calculator value of $\sqrt[3]{26}$.

5. Use the linear approximation to estimate $\sqrt[4]{16.2}$ with a rational number. Compare to the calculator value of $\sqrt[4]{16.2}$.

6. The cost of producing 10 items is \$200 and the cost of each new item produced is \$15. Approximate the cost of producing 12 items.

7. The number of bacteria five hours after the start of experiment is 2000 and the number is increasing by 100 bacteria per hour. Approximate the number of bacteria five and a half hours after the start of experiment.

Solutions.

1. (a) Let $f(x) = \sin x$ so that $f'(x) = \cos x$. Thus $f(0) = 0$ and $f'(0) = \cos 0 = 1$. The linear approximation $f(x) \approx f(a) + f'(a)(x - a)$ for $a = 0$ gives you that

$$\sin x \approx 0 + 1(x - 0) = x.$$

   Note that this answer also indicates that the line x is tangent to the graph of $\sin x$ at $x = 0$.

When $x = 0.1$ this approximation gives you $\sin 0.1 \approx 0.1 = \frac{1}{10}$. To check out the accuracy, you can compare with the calculator value $\sin 0.1 \approx 0.998\ldots$

(b) Let $f(x) = \ln(1+3x)$ so that $f'(x) = \frac{3}{1+3x}$. Thus $f(0) = 0$ and $f'(0) = 3$. The linear approximation $f(x) \approx f(a)+f'(a)(x-a)$ for $a = 0$ gives you that $\ln(1+3x) \approx 0+3(x-0) = 3x$. So $3x$ is tangent to the graph of $\ln(1+3x)$ at $x = 0$.

Note that when $y = \ln 0.7 = \ln(1-3(0.1))$, $x = -0.1$. The linear approximation with $x = -0.1$ is $\ln 0.7 \approx 3(-0.1) = -0.3 = \frac{-3}{10}$. To check out the accuracy, you can compare with the calculator value $\ln 0.7 = -0.367\ldots$

2. $f(2) = 5$ and $f'(2) = 3 \Rightarrow f(x) \approx f(2) + f'(2)(x-2) = 5 + 3(x-2)$ or $3x - 1$. Thus $f(2.1) \approx 5 + 3(2.1-2) = 5 + 0.3 = 5.3$ and $f(1.9) \approx 5 + 3(1.9-2) = 5 - 0.3 = 4.7$.

3. $f(1) = 1$ and $f'(1) = -2 \Rightarrow f(x) \approx f(1) + f'(1)(x-1) = 1 - 2(x-1)$ or $2x + 3$. Thus $f(1.01) \approx 1 - 2(1.01-1) = 1 - 0.02 = 0.98$.

4. To approximate $\sqrt[3]{26}$, consider the function $f(x) = \sqrt[3]{x} = x^{1/3}$ and its linear approximation at $a = 27$ (since $\sqrt[3]{27} = 3$ is easy to determine and 26 is relatively close to 27. In this case, $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$. Thus $f(27) = 3$ and $f'(27) = \frac{1}{3}\frac{1}{3^2} = \frac{1}{27}$ so that the linear approximation is $x^{1/3} \approx 3 + \frac{1}{27}(x-27)$. When $x = 26$, the linear approximation is $26^{1/3} \approx 3 - \frac{1}{27} = \frac{80}{27} \approx 2.962963$. The calculator value is $2.9624960\ldots$

5. To approximate $\sqrt[4]{16.2}$, consider the function $f(x) = \sqrt[4]{x} = x^{1/4}$ and its linear approximation at $a = 16$ (since $\sqrt[4]{16} = 2$ is easy to determine and 16.2 is relatively close to 16. In this case, $f(x) = x^{1/4} \Rightarrow f'(x) = \frac{1}{4}x^{-3/4}$. Thus $f(16) = 2$ and $f'(16) = \frac{1}{4}\frac{1}{2^3} = \frac{1}{32}$ so that the linear approximation is $x^{1/4} \approx 2 + \frac{1}{32}(x-16)$. When $x = 16.2$, the linear approximation is $16.2^{1/4} \approx 2 + \frac{1}{32}(0.2) = 2 + \frac{1}{160} = \frac{321}{160} = 2.00625$. The calculator value is $2.00622\ldots$

6. If x denotes the number of items produced and $f(x)$ denotes the cost of producing x items, then $f(10) = 200$ and $f'(10) = 15$. Thus, the linear approximation is $f(x) \approx f(10)+f'(10)(x-10) = 200+15(x-10)$. When $x = 12$ this approximation gives you $f(12) \approx 200+15(2) = 230$ dollars.

7. If x denotes the number of hours after the start of experiment and $f(x)$ denotes the number of bacteria at that time, then $f(5) = 2000$ and $f'(5) = 100$. Thus, the linear approximation is $f(x) \approx f(5) + f'(5)(x-5) = 2000 + 100(x-5)$. When $x = 5.5$ this approximation gives you $f(5.5) \approx 2000 + 100(0.5) = 2050$ bacteria.