Calculus 1
Lia Vas

Review for the Final Exam

1. **Integrals.** Evaluate the following integrals.

   (a) $\int(\sqrt{x} - \frac{4}{x^2})\, dx$

   (b) $\int(2\sqrt[3]{x} + \frac{x^2}{4})\, dx$

   (c) $\int(3x+5)^6\, dx$

   (d) $\int \frac{x^2}{x^3+1}\, dx$

   (e) $\int \frac{x}{\sqrt{1-9x^2}}\, dx$

   (f) $\int \frac{x-1}{x^2}\, dx$

   (g) $\int xe^{x^2+1}\, dx$

   (h) $\int 2^{3x+1}\, dx$

   (i) $\int x\sqrt{x-3}\, dx$

   (j) $\int x^3\sqrt{1+x^2}\, dx$

   (k) $\int bxe^{ax^2+1}\, dx$ where a and b are arbitrary constants.

   (l) $\int \frac{1}{ax+b}\, dx$ where a and b are arbitrary constants.

   (m) $\int(e^{2x} + e^{-2x})\, dx$

   (n) $\int \cos(5x+1)\, dx$

   (o) $\int(2+\sin 2x)\, dx$

   (p) Find the function $f(x)$ which has the derivative $f'(x) = \frac{10}{\sqrt{4x+1}}$ and satisfies the condition $f(0) = 3$.

   (q) Find the function $f(x)$ which has the derivative $f'(x) = x^2\sqrt{4x^4+9x^2}$ and and satisfies the condition $f(0) = -2$.

2. **Area.** Find the following areas.

   (a) Area between $f(x) = x^2 - 2x$ and x-axis for $1 < x < 3$

   (b) Area between $f(x) = 2\sqrt{x} - 4$ and x-axis for $0 < x < 9$

   (c) Area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ for $1/2 < x < 1$

   (d) Area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ for $1 < x < 4$

   (e) Area between $y = 4 - x^2$ and $y = -x + 2$

   (f) Area between $y = 2x$ and $y = x^2 - 4x$

   (g) Area between $y = 4x^2$ and $y = x^2 + 3$

   (h) Area between $y = x^3$ and $y = 3x^2 - 2x$

   (i) Area between $y = x^3$ and $y = x$

   (j) Area between $y = -x^3$ and $y = 12x - 7x^2$

3. **Approximate Integration.** Approximate the following integral using the Left and Right Sums Program to the first two nonzero digits.

   (a) $\displaystyle\int_0^2 \ln(x^2+1)\, dx$

   (b) $\displaystyle\int_1^3 \frac{e^{2x}}{x}\, dx$

4. **Applications of Integrals.**

(a) A stone is being thrown up in the air with initial velocity of 5 m/s. Determine the time the object hits the ground and find the speed at the time of the impact. Determine also the maximal height it reaches. You can neglect the drag force and assume that the gravitational force is the only force that acts on an object producing the constant acceleration of 9.8 m/s^2.

(b) A chemical reaction produces a compound X with a rate of 23, 19, 12, 11, 9, 5, 2 liters per second at time intervals spaced by 1 second. Approximate the total volume of the compound X produced in the 6 seconds for which the rate is given.

(c) The size of a certain bacteria culture grows at a rate of $f(t) = te^{t/2}$ milligrams per hour. Use your calculator program to approximate the total change in the bacteria size after the first 3 hours to the first two nonzero digits.

(d) Approximate the area of the lake using the shown measurements of its width which were taken 50 feet apart.

0	88	110	145	180	138	129	93	84	0

(e) From past records, a botanist knows that a certain species of tree has a rate of growth that can be modeled by $f(t) = \frac{2}{\sqrt{t}}$, $1 \leq t \leq 4$, where t is the age of the tree in years and $f(t)$ is the growth rate in feet per year. Determine how much did the tree grow from the time when it was a year old to the time it was four years old.

(f) Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{\sqrt{t^2+9}}$ where t is the time in seconds and v is the velocity in feet per second. Determine the total movement of the object between 3 and 5 seconds.

(g) The rate of change in the U.S. population can be modeled by $g(x) = 1.03e^{0.013t}$, $0 \leq t \leq 100$ where t represents the number of years since 1900 and g represents the rate of change in population measured in millions per year. Determine the total increase in the U.S. population from 1900 to 1950.

(h) Geologists estimate that an oil field will produce oil at a rate given by $f(t) = 600e^{-0.1t}$ thousand barrels per month, t months into production. Estimate the total production for the first year of operation. Round to the nearest whole number.

(i) Breathing is cyclic and a full respiratory cycle takes about 5 seconds. The function $f(t) = \frac{1}{2}\sin\frac{2\pi t}{5}$ in liters per second has often been used to model the rate of air flow into the lungs at time t. Find the volume of inhaled air in the lungs in one respiratory cycle.

(j) Pollution enters a lake at the rate $f(t) = 150 - 0.2e^{t/2}$ g/hour. Meanwhile, the pollution filter removes the pollution at the rate of $g(t) = 0.3e^{t/2}$ g/hour. (i) Find the time when the rate of pollution entering is the same as the rate pollution leaving the lake and the amount of pollution at that time. (ii) If the initial amount of pollution is 500 g, determine the function computing the total amount of pollution at time t. Then find the time when the pollution is completely removed from the lake using your calculator.

5. **Limits.** Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2-3x+2}$

(b) $\displaystyle\lim_{x\to\infty} \frac{x-1}{x^2-3x+2}$

(c) $\displaystyle\lim_{x\to 0^+} \frac{x+2}{x(x-2)}$ (d) $\displaystyle\lim_{x\to\infty} \frac{x+2}{x(x-2)}$

(e) Let $f(x) = \begin{cases} -x-1 & x<-1 \\ 1-x & -1 \le x < 1 \\ \sqrt{x-1} & x \ge 1 \end{cases}$ Evaluate the following:

$$\lim_{x\to -1^-} f(x) \qquad \lim_{x\to -1^+} f(x) \qquad \lim_{x\to -1} f(x) \qquad f(-1) \qquad \lim_{x\to 1} f(x) \qquad \lim_{x\to 0} f(x)$$

(f) $\lim_{x\to -\infty} 3^{\frac{4}{x-2}} - 5$ (g) $\lim_{x\to 2^-} 3^{\frac{4}{x-2}} - 5$ (h) $\lim_{x\to 2^+} 3^{\frac{4}{x-2}} - 5$

(i) $\lim_{x\to -1^+} \ln(x+1)+3$ (u) $\lim_{x\to\infty} \ln(x+1)+3$ (j) $\lim_{x\to\infty} \ln(x+1) - \ln(2x+3)$

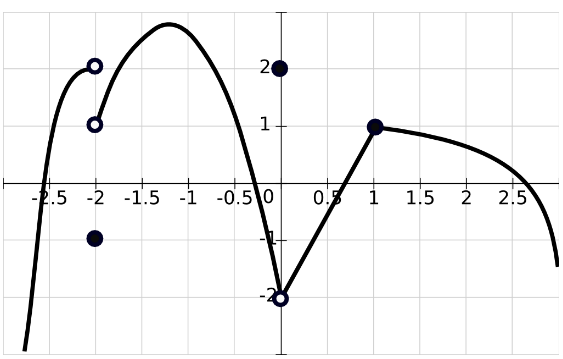
(k) $\lim_{x\to\infty} \frac{\cos x - 1}{x^2}$ (l) $\lim_{x\to\infty} \sin\frac{x^2-x}{3+2x^2}$ (m) $\lim_{x\to\infty} \cos\frac{x-1}{x^2}$

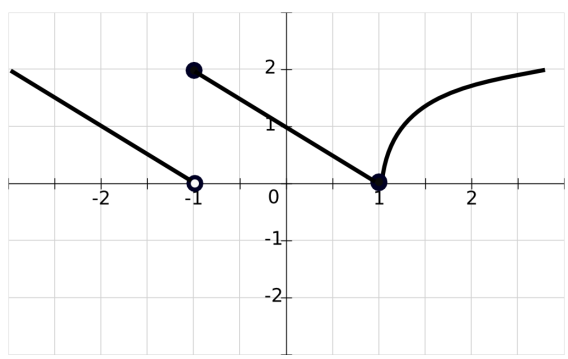
6. **Asymptotes.** Find the horizontal and vertical asymptotes (if any) of the following functions.

(a) $f(x) = \dfrac{x^2-2x-8}{x^2-x-6}$ (b) $f(x) = \dfrac{x^3+1}{x(3-x)}$

7. **Continuity and Differentiability.** Discuss the continuity and differentiability of the following functions at every point.

(a) $f(x) = 2 - 3x^{1/5}$ (b) The function given by the graph on the left. (c) The function given by the graph on the right.





8. **Derivative definition.** Find the derivative of the following functions at a given point using the definition of derivative at a point.

(a) $f(x) = x^2 - 3x,\ x = 2$ (b) $f(x) = \frac{1}{x}$, any x.

9. **Limit and Derivative Applications.**

(a) The function $B(t) = \frac{2\cdot 10^7}{1+7e^{-3t/10}}$ models the biomass (total mass of the members of the population) in kilograms of a Pacific halibut fishery after t years. Determine the biomass in the long run.

(b) The distance in miles a traveling car passes after it started moving is represented in the following table as a function of time in hours.

time (hours)	0	0.5	1	1.5	2	2.5
distance (miles)	0	30	52	52	76	104

(i) Find the average velocity of the vehicle between the first and the second hour.

(ii) Estimate the velocity two hours after the vehicle started moving.

(iii) Based on the information given, estimate the initial velocity of the vehicle.

(iv) Based on the information given, what can you say about the movement of the vehicle between the first hour and the first hour and a half?

(c) A company determines that its cost function is $C(x) = 1000 + 35x - .01x^2$, $0 \leq x \leq 300$, where x is the number of items produced and $C(x)$ is the cost of producing x items in dollars. Find the average rate of change in cost when x is changing from 100 to 150. Then, find the instantaneous rate of change in cost when producing 200 units and estimate the cost of producing 201 items.

(d) The concentration of a certain medication in a patient's bloodstream (in mg per cm^3) is given by $C(t) = \frac{5t}{t^2+4}$, where t is the number of hours after the medication has been administered. (i) Determine the concentration 3 hours after the medication is administered. (ii) Determine how fast the concentration changes 3 hours after the medication is administered. (iii) Determine how fast the concentration changes on average between 2nd and 4th hour.

(e) The concentration of pollutants (in grams per liter) in a river is approximated by $C(x) = .04e^{-4x}$ where x is the number of miles downstream from a place where the measurements are taken. (i) Determine the initial pollution and the pollution 2 miles downstream. (ii) Determine how much the concentration changes on average within the first two miles. (iii) Determine how fast the concentration changes 2 miles downstream.

(f) The body mass index (BMI) is a number obtained as $BMI = \frac{703w}{h^2}$ where w is the weight in pounds and h is the height in inches. For a 125-lb female that is now 65 inches tall but growing, calculate how fast is BMI changing with each new inch. Explain the meaning of the answer.

(g) An arrow has been shot in the air and its height above the ground is described by the formula $s(t) = 24t - 4.9t^2$ where t is in seconds and s is in meters. (i) Determine the acceleration, graph the height, velocity and acceleration on the same plot and determine when the arrow speeds up and when it slows down by discussing the sign of the velocity and acceleration. (ii) Determine the time the arrow is at the highest distance from the ground. (iii) Determine the time the arrow falls down to the ground and its speed at the time of the impact.

10. **Finding Derivative.** Find the derivative for the given function. Assume that $f(x)$ is a function differentiable for every value of x in problems (u)–(z).

(a) $y = \log_2(x^2 + 7x)$

(b) $x^2 + xy^4 = 6$

(c) $y = (3x+2)^{2x-1}$

(d) $y = x^2 \cos(x^2)$

(e) $y = \frac{e^{2x}+e^{-2x}}{x^2}$

(f) $x^3 + 12xy = y^3$

(g) $y = (\ln x)^x$

(h) $y = \frac{(x^2+3)^4}{(3x^2+1)^5}$

(i) $y = 3^{2x^2+5}$

(j) $y = (2x + e^{x^2})^4$

(k) $y = \ln(5x - e^{5x})$

(l) $xe^y + x^2 = y^2$

(m) If $F(x) = (x^5+1)f(x)$, $f(0) = 0$ and $f'(0) = 2$, determine $F'(0)$.

(n) If $F(x) = \ln(f(x)+1)$, $f(1) = 0$ and $f'(1) = 1$, determine $F'(1)$.

(o) If $f(x)$ has the inverse and $f(3) = 2$ and $f'(3) = 6$, find $(f^{-1})'(2)$.

11. **Tangent Line.** Find an equation of the line tangent to the graph of the given equation at the indicated point.

(a) $x^2 + y^2 = 13, \quad (3,2)$

(b) $x \ln y = 2x^3 - 2y, \quad (1,1)$

(c) $y = \ln\sqrt{2x-1}, \quad x = 1$

(d) $x^2 + y^2 = e^y, \quad (1,0)$

12. **Checking solutions of differential equations.**

(a) Show that $y = ce^{2x}$ is a solution of the differential equation $y'' - 3y' + 2y = 0$ for every value of the constant c.

(b) Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

13. **Linear Approximations.**

(a) If $f(2) = 5$ and $f'(2) = 3$, approximate $f(1.9)$.

(b) Use the linear approximation to estimate $\sqrt[3]{26}$. Compare to the calculator value of $\sqrt[3]{26}$.

(c) The number of bacteria 5 hours after the start of experiment is 2000. The number is increasing by 100 bacteria per hour. Approximate the number of bacteria 5.5 hours after the start of experiment.

(d) The profit P of a company depends on the number of items x produced. The production level x depends on the time t (measured in years). Assume that the profit increases by \$200 with each new item produced and that the production level increases by 150 items each year. (i) Determine the rate of increase of the profit per year. (ii) If the company is presently making a profit of \$800,000, approximate the profit in four year time.

14. **Related Rates.**

(a) A 20-foot ladder is leaning against the wall. If the base of the ladder is sliding away from the wall at the rate of 3 feet per second, find the rate at which the top of the ladder is sliding down when the top of the ladder is 8 feet from the ground.

(b) A 6-feet-tall man walks at the rate of 5 feet per second towards a 24-feet-tall street lamp. Determine how fast is the tip of man's shadow moving along the ground.

(c) A conical tank of height 2 meters is full of water. The radius of the surface is 1 meter. If the water evaporates at the rate of 30 centimeters cubic per day, determine the rate at which the water level decreases when the water is 0.5 meters deep. Discuss if this rate is increasing or decreasing as the depth of the water becomes smaller. Recall that the volume of a cone of height h with the radius of the base r is given by $V = \frac{1}{3}r^2 h\pi$.

(d) Two resistors with resistances R_1 and R_2 are connected in parallel into an electrical circuit. The total resistance R in ohms is computed by the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are increasing by 0.25 ohms per second, determine how fast is R changing when $R_1 = 75$ and $R_2 = 100$ ohms.

15. **Derivative Tests. Graphical Analysis.** In problems (a) to (c), find the intervals where the function is increasing and where it is decreasing; Find the intervals where the function is concave up and where it is concave down. Find the relative minimum, relative maximum and the inflection points. Graph the given function. Choose the appropriate scale to see the entire graph with all the relevant points (intercepts, extreme and inflection points) on it.

(a) $f(x) = \frac{x^3}{3} + x^2 - 15x + 3$ (b) $f(x) = \frac{1}{x} + \frac{x}{16}$ (c) $f(x) = \frac{\ln x + x}{x}$

(d) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing. Use the First or the Second Derivative Test to find the relative minimum and maximum values (if any). (i) $f(x) = \frac{2x}{x^2+4}$ (ii) $f(x) = e^x(x^2 - x - 5)$

(e) Given the derivatives f' and f'' of the function f, determine the intervals on which $f(x)$ increases/decreases, the intervals on which the function is concave up/down and x values in which the function has maximum, minimum and inflection.

(i) $f'(x) = \frac{(x-6)(x-1)}{(x+3)}$, $f''(x) = \frac{(x+9)(x-3)}{(x+3)^2}$, $f(-3)$ not defined.

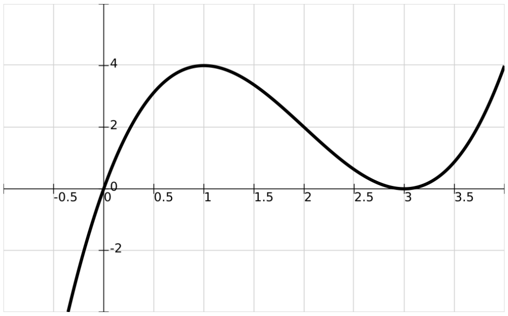
(ii) $f'(x) = \frac{(x-8)(x+1)}{(x+4)}$, $f''(x) = \frac{(x+10)(x-2)}{(x+4)^2}$ $f(-4)$ not defined.

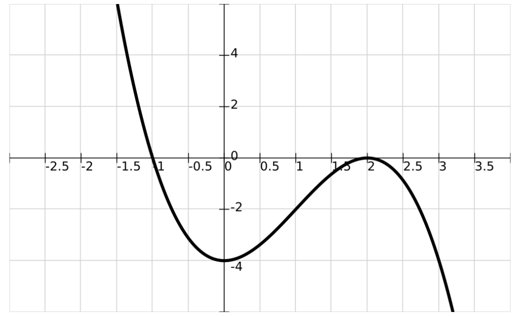
16. **Graphical Analysis continued.** Given the properties of a function $f(x)$ in problems (a)–(b) below, determine the minimum and maximum values of $f(x)$, inflection points, the intervals on which f is concave up/down, and sketch the graph of one possible function with the given properties.

(a) $f(3) = 1$, $f(-3) = -1$, $f(0) = 0$, $f'(3) = 0$, $f'(-3) = 0$, $f''(x) > 0$ on $(-\infty, 0)$, and $''(x) < 0$ on $(0, \infty)$.

(b) $f(-2) = 2$, $f(0) = -2$, f has a vertical asymptote at $x = 2$, $f'(-2) = 0$, $f'(0) = 0$, f' is not changing the sign at $x = 2$, $f''(x) > 0$ on (-1, 2), and $f''(x) < 0$ on $(-\infty, -1)$ and $(2, \infty)$.

(c) and (d) Assuming that the graph below is the graph of the derivative of a function $f(x)$, determine the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down, determine the minimum and maximum values of $f(x)$, as well as the inflection points. Sketch a graph of one possible function with the given derivative.





graphs for problems (c) and (d)

17. **Absolute Extrema.** Find the absolute minimum and maximum of each function on the indicated interval. You can use your calculator to find the zeros of the first derivative if necessary.

    (a) $f(x) = 3x^4 + 4x^3 - 36x^2 + 1, \ \ [-1, 4]$

    (b) $f(x) = x^4 - 15x^2 - 10x + 24, \ \ [-3, 3]$

18. **Optimization Problems.**

    (a) Consider the drug concentration function $C(t) = 2te^{-.4t}$ where C (in μg/cm^3) is the concentration of a drug in the body at time t hours after the drug was administered. (i) Find the time intervals when the concentration is increasing/decreasing. (ii) Determine the time when the concentration decrease is the largest and the value of that largest rate.

    (b) The function $B(t) = 5 - \frac{1}{9}\sqrt[3]{(8-3t)^5}$ models the biomass (total mass of the members of the population) in kilograms of a mice population after t months. (i) Determine when the population increases at a smallest rate. Determine also that rate and the biomass at that time. (ii) Determine when the population is smallest and when it is the largest between 3 and 6 months after it started being monitored.

    (c) In a physics experiment, temperature T (in Fahrenheit) and pressure P (in kilo Pascals) have a constant product of 5000 and the function $F = T^2 + 50P$ is being monitored. Determine the temperature T and pressure P that minimize the function F.

    (d) A soup manufacturer intents to sell the product in a cylindrical can that should contain half a liter of soup. Determine the dimensions of the can which minimize the amount of the material used. Recall that a liter corresponds to decimeter cubic and express your answer in centimeters.

    (e) Find the point on the parabola $y^2 = 2x - 2$ which is closest to the point $(2, 4)$.

    (f) Find the dimensions of a rectangle of the largest area which has the base on x-axis and the opposite two vertices on the parabola $y = 12 - x^2$.

Review for Final Exam – Solutions

More detailed solutions of the problems can be found on the class handouts.

1. Integrals. (a) $\frac{2}{3}x^{3/2}+\frac{4}{x}+c$ (b) $\frac{3}{2}x^{4/3}+\frac{x^3}{12}+c$ (c) $\frac{1}{21}(3x+5)^7+c$ (d) $\frac{1}{3}\ln|x^3+1|+c$ (e) $-\frac{1}{9}\sqrt{1-9x^2}+c$ (f) $\ln|x|+\frac{1}{x}+c$ (g) $\frac{1}{2}e^{x^2+1}+c$ (h) $\frac{1}{3\ln 2}2^{3x+1}+c$ (i) Use $u=x-3$. The integral becomes $\int(u+3)\sqrt{u}du=\int(u^{3/2}+3u^{1/2})du=\frac{2}{5}u^{5/2}+3\frac{2}{3}u^{3/2}+c=\frac{2}{5}(x-3)^{5/2}+2(x-3)^{3/2}+c.$

   (j) $u=1+x^2$. The integral becomes $\frac{1}{2}\int x^2\sqrt{u}\,du$. Use the formula $u=x^2+1$ to express x^2 in terms of u as $x^2=u-1$. The integral becomes $\frac{1}{2}\int(u-1)\sqrt{u}du=\frac{1}{2}\int(u^{3/2}-u^{1/2})du=\frac{1}{2}(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2})+c=\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+c.$ (k) $\frac{b}{2a}e^{ax^2+1}+c.$ (l) $\frac{1}{a}\ln|ax+b|+c.$ (m) $\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+c$ (n) $\frac{1}{5}\sin(5x+1)+c$ (o) $2x-\frac{1}{2}\cos 2x+c$ (p) $f(x)=5(4x+1)^{1/2}-2$ (q) $f(x)=\int x^2\sqrt{4x^4+9x^2}dx$. Use $u=4x^2+9$. $f(x)=\int x^3\sqrt{u}\frac{du}{8x}=\frac{1}{8}\int x^2\sqrt{u}\,du.$ From $u=4x^2+9$, $x^2=\frac{1}{4}(u-9)$ Thus $f(x)=\frac{1}{32}\int(u-9)\sqrt{u}du=\frac{1}{32}\int(u^{3/2}-9u^{1/2})du=\frac{1}{32}(\frac{2}{5}u^{5/2}-9\frac{2}{3}u^{3/2})+c=\frac{1}{80}(4x^2+9)^{5/2}-\frac{3}{16}(4x^2+9)^{3/2}+c$. $f(0)=-2\Rightarrow c=\frac{1}{40}$ resulting in $f(x)=\frac{1}{80}(4x^2+9)^{5/2}-\frac{3}{16}(4x^2+9)^{3/2}+\frac{1}{40}$.

2. Area. (a) 2 (b) 32/3 (c) 2.61 (d) 1 (e) 9/2 (f) 36 (g) 4 (h) 1/2 (i) 1/2 (j) 11.25+.583=11.83

3. Approximate Integration. (a) With $n=100$, left sum = right sum =1.4 (b) With $n=300$, left sum = right sum = 81

4. Applications of Integrals. (a) $a(t)=v'(t)=-9.8\Rightarrow v(t)=\int -9.8dt=-9.8t+c$. $v(0)=5\Rightarrow 5=-9.8(0)+c\Rightarrow c=5$ and so $v(t)=s'(t)=-9.8t+5$. $s(t)=\int(-9.8t+5)dt=-4.9t^2+5t+c$. $s(0)=0\Rightarrow c=0$ and so $s(t)=-4.9t^2+5t$. The object hits the ground when $s(t)=0\Rightarrow t=1.02$ seconds. The velocity at that time is $v(1.02)=-5$ m/s. To find the maximal height, solve $v(t)=-9.8t+5=0$. Get $t=\frac{5}{9.8}=0.51$ second and the maximal height $s(\frac{5}{9.8})\approx 1.28$ meters.

   (b) Left =upper estimate = 79 liters, Right = lower estimate = 58 liters. (c) Size = $\int_0^3$ rate $dt=\int_0^3 te^{t/2}dt$. With $n=100$, get Left=Right=13 milligrams. (d) Left = Right = 48350 square feet.

   (e) $\int_1^4\frac{2}{\sqrt{t}}dt$. Obtain 4 feet. (f) $s(5)-s(3)=\int_3^5 v(t)dt$. Obtain 1.588 ft. (g) 72.54 millions. (h) $4192.834\approx 4193$ thousands of barrels. (i) $\frac{5}{2\pi}\approx 0.796$ liters.

   (j) (i) $150-0.2e^{t/2}=0.3e^{t/2}\Rightarrow t=2\ln 300\approx 11.41$ hours. The amount of pollution at $t\approx 11.41$ is $\int_0^{11.41}\left(150-0.2e^{t/2}-0.3e^{t/2}\right)dt=1412.13$ grams. (ii) The amount of pollution $A(t)=150t-e^{t/2}+c$. $A(0)=500\Rightarrow c=501$. So $A(t)=150t-e^{t/2}+501$. Using calculator to solve $A(t)=0$, find $t\approx 15.94$ hours.

5. Limits. (a) -1 (b) 0 (c) $-\infty$ (d) 0 (e) $0,2,$ doesn't exist, $2,0,1.$ (f) -4 (g) -5 (h) ∞ (i) $-\infty$ (j) ∞ (k) Note that $\ln(x+1)-\ln(2x+3)=\ln\frac{x+1}{2x+3}$ The limit is $\ln\frac{1}{2}=-.693$. (l) Use the Squeeze Theorem to get 0. (m) $\sin\frac{1}{2}=.479$ (n) $\cos 0=1$

6. Asymptotes. (a) $x=3$ is the only vertical asymptote and $y=1$ is a horizontal asymptote. (b) $x=3$ and $x=0$ are vertical asymptotes and $f(x)$ does not have a horizontal asymptote.

7. Continuity and Differentiability. (a) $f(x)$ is continuous at every point and differentiable for every x except $x = 0$ because of a vertical tangent. (b) $f(x)$ is differentiable at every point different from -2, 0 and 1. At $x = -2$ and $x = 0$ the function has a break and a hole so it is not continuous and hence not differentiable. At $x = 1$ the function is not differentiable since there is a corner in the graph but it is continuous. (c) The function is differentiable at every point different from -1 and 1. At $x = -1$ there is a break so the function is not differentiable since it is not continuous. At $x = 1$ the function is not differentiable since there is a corner in the graph but it is continuous.

8. Derivative Definition. (a) 1 (b) $f'(x) = \frac{-1}{x^2}$

9. Limit and Derivative Applications. (a) $2 \cdot 10^7$ kg. (b) (i) 24 miles per hour. (ii) 52 miles per hour. (iii) 60 miles per hour. (iv) Between the first hour and the first hour and a half.

   (c) When production changes from 100 to 150 items produced, the cost increased at an average rate of \$32.5 per item produced. When producing 200 items, the cost is increasing at a rate of about \$31 per item produced. $C(201) \approx 7631$.

   (d) (i) 1.15 mg/cm^3 (ii) $C'(3) = -.148$ thus, the concentration is decreasing by .148 mg/cm^3 per hour. (iii) $\frac{1-1.25}{4-2} = -.124$, thus the concentration is decreasing on average by .124 mg/cm^3 per hour between hour 2 and 4.

   (e) (i) $C(0) = .04$ and $C(2) = 1.3 \cdot 10^{-5}$ grams per liter (ii) $-.01999 \approx -.02$. Thus the concentration is decreasing on average by .02 grams per liter per mile during the first two miles. (iii) $C'(2) = -5.37 \cdot 10^{-5}$, thus the concentration is decreasing by .0000537 grams per liter per mile 2 miles downstream.

   (f) The value of the derivative of $\frac{703(125)}{h^2}$ at $h = 65$ is $-.6399 \approx -.64$. Thus, the BMI is decreasing by .64 per inch.

   (g) (i) $s''(t) = a(t) = -9.8$. The arrow slows down in the first 2.45 seconds and it speeds up between 2.45 and 4.9 seconds. (ii) 2.45 seconds. (iii) 4.9 seconds. 24 meters per second.

10. Derivative.

   (a) $\frac{2x+7}{\ln 2(x^2+7x)}$

   (b) Implicit differentiation. $-\frac{2x+y^4}{4xy^3}$

   (c) Use logarithmic differentiation $\ln y = \ln(3x+2)^{2x-1} = (2x-1)\ln(3x+1) \Rightarrow \frac{1}{y}y' = 2\ln(3x+1) + \frac{3(2x-1)}{3x+2} \Rightarrow y' = \left(2\ln(3x+1) + \frac{3(2x-1)}{3x+2}\right) y \Rightarrow y' = (2\ln(3x+2) + \frac{3(2x-1)}{3x+2})(3x+2)^{2x-1}$.

   (d) $2x\cos x^2 - 2x^3 \sin x^2$

   (e) $\frac{(2e^{2x}-2e^{-2x})x^2-2x(e^{2x}+e^{-2x})}{x^4}$

   (f) Implicit differentiation. $\frac{x^2+4y}{y^2-4x}$

   (g) Use logarithmic differentiation $\ln y = \ln(\ln x)^x = x\ln(\ln x) \Rightarrow \frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x}\frac{1}{x}x \Rightarrow y' = \left(\ln(\ln x) + \frac{1}{\ln x}\right) y \Rightarrow y' = (\ln(\ln x) + \frac{1}{\ln x})(\ln x)^x$.

   (h) $\frac{4(x^2+3)^3 2x(3x^2+1)^5 - 5(3x^2+1)^4 6x(x^2+3)^4}{(3x^2+1)^{10}}$

(i) $3^{2x^2+5} \cdot \ln 3 \cdot 4x$

(j) $4(2x + e^{x^2})^3 \cdot (2 + e^{x^2} 2x)$

(k) $\frac{5-5e^{5x}}{5x-e^{5x}}$

(l) Implicit differentiation. $\frac{e^y+2x}{2y-xe^y}$

(m) $F'(x) = 5x^4 f(x) + f'(x)(x^5 + 1) = f(x) + xf'(x) \Rightarrow F'(0) = 5(0)^4 f(0) + f'(0)(0^5 + 1) = 0 + 2(1) = 2.$

(n) $F'(x) = \frac{f'(x)}{f(x)+1}$. Since $f(1) = 0$ and $f'(1) = 1$, $F'(1) = \frac{1}{0+1} = 1$.

(o) Since $f(3) = 2$, $(f^{-1})'(2) = \frac{1}{f'(3)}$. Then since $f'(3) = 6$, $(f^{-1})'(2) = \frac{1}{f'(3)} = \frac{1}{6}$.

11. Tangent line. (a) $y = -3/2x + 13/2$ (b) $y = 2x - 1$ (c) $y = x - 1$ (d) $y = 2x - 2$

12. Checking solutions of differential equations. (a) $y = ce^{2x} \Rightarrow y' = 2ce^{2x} \Rightarrow y'' = 4ce^{2x}$. Plug these into the equation $y'' - 3y' + 2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4-6+2)ce^{2x} = 0 \Rightarrow 0 = 0$. Thus y is a solution of the equation.

    (b) $y = Ae^{3x} \Rightarrow y' = 3Ae^{3x} \Rightarrow y'' = 9Ae^{3x}$. Plug these into the equation $y'' - 3y' + 2y = 6e^{3x} \Rightarrow 9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3$. Thus, $y = 3e^{3x}$ is a solution.

13. Linear Approximations. (a) 4.7 (b) 2.962963. Calculator value: 2.962496. (c) 2050 bacteria.
    (d) (i) $\frac{dP}{dx} = 200$ dollars per item and $\frac{dx}{dt} = 150$ items per year. So, $\frac{dP}{dt} = \frac{dP}{dx}\frac{dx}{dt} = 200 \cdot 150 = 30,000$ dollars per year. (ii) $800,000 + 4 \cdot 30,000 = 920,000$ dollars.

14. Related Rates. (a) Decreasing by 6.87 ft per sec. (b) Decreasing by $\frac{5}{3} \approx 1.67$ ft per sec. (c) Decreasing by 0.015 cm per day. (d) $\frac{1}{336} \approx 0.003$ ohms per second.

15. Derivatives and Graphs.

    (a) f is increasing for $x < -5$ and $x > 3$ and decreasing for $-5 < x < 3$, concave up for $x > -1$ and concave down for $x < -1$. The relative minimum is (3, -24), the relative maximum is (-5, 61.33) and the inflection point is (-1, 18.67).

    (b) f is increasing for $x < -4$ and $x > 4$, decreasing for $-4 < x < 0$ and $0 < x < 4$, concave up for $x > 0$, and concave down for $x < 0$. The relative minimum is (4, 1/2), the relative maximum is (-4, -1/2) and there are no inflection points.

    (c) f is increasing for $0 < x < e$, decreasing for $x > e$, concave up for $x > e^{3/2}$, and concave down for $0 < x < e^{3/2}$. There is no relative minimum, the relative maximum is $(e, \frac{1+e}{e}) \approx (2.72, 1.37)$ and the inflection point is $(e^{3/2}, f(e^{3/2})) \approx (4.48, 1.33)$.

    (d) (i) f is increasing for $-2 < x < 2$; decreasing for $x < -2$ and $x > 2$. At $x = 2$ there is a maximum; at $x = -2$ there is a minimum. Max. value 1/2. Min. value -1/2.

    (ii) f is increasing for $x > 2$ and $x < -3$; decreasing for $-3 < x < 2$. At $x = -3$ there is a maximum; at $x = 2$ there is a minimum. Max. value .348. Min. value -22.17.

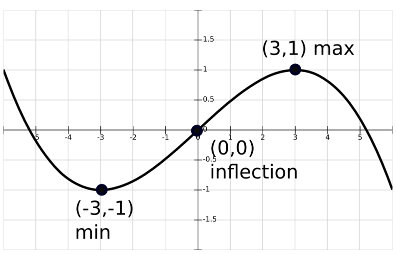
(e) (i) f is increasing for $x > 6$ and $-3 < x < 1$ and decreasing for $x < -3$ and $1 < x < 6$. At $x = 1$ there is a maximum, at $x = 6$ a minimum, and no extreme value at $x = -3$. f is concave up for $x > 3$ and $x < -9$, and concave down for $-9 < x < 3$. There are inflection points at $x = 3$ and $x = -9$ and no inflection point at $x = -3$.

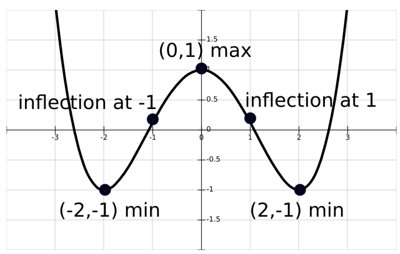
(ii) f is increasing for $x > 8$ and $-4 < x < -1$ and decreasing for $x < -4$ and $-1 < x < 8$. At $x = -1$ there is a maximum and, at $x = 8$ a minimum, and no extreme value at $x = -4$. f is concave up for $x > 2$ and $x < -10$, and concave down for $-10 < x < 2$. There are inflection points at $x = 2$ and $x = -10$ and no inflection point at $x = -4$.

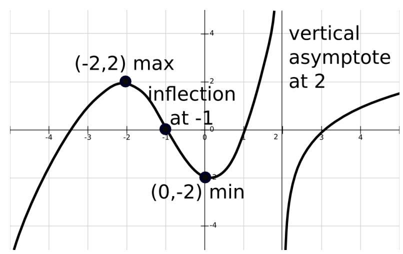
16. Graphical Analysis continued.

(a) f passes (3,1), (-3,-1) and (0,0). 3 and -3 are critical points. Since $f''(x) > 0$ on $(-\infty, 0)$, $f''(-3) > 0$ so there is a minimum at -3. Since $f''(x) < 0$ on $(0, \infty)$, $f''(3) < 0$ so there is a maximum at 3. f is concave up on $(-\infty, 0)$, concave down on $(0, \infty)$, and (0,0) is an inflection point.

(b) f passes (-2,2) and (0,-2) and has a vertical asymptote at $x = 2$. 2, -2 and 0 are critical points. $f''(-2) < 0$ so there is a maximum value at -2. $f''(0) > 0$ so there is a minimum value at 0. f is concave up on (-1, 2) and concave down on $(-\infty, 1)$ and $(2, \infty)$. There is an inflection point at -1. There is neither an extreme value nor inflection point at 2.

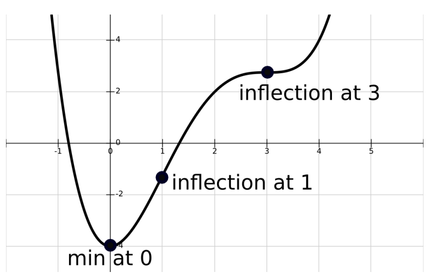


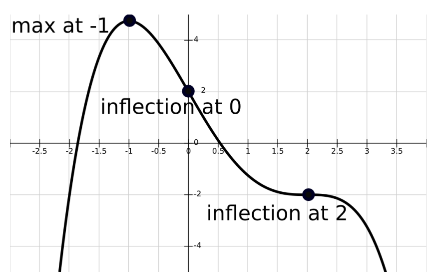




(c) The critical points are 0 and 3. f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. f' changes from negative to positive at 0, so there is a minimum at 0. f is concave up on $(-\infty, 1)$ and on $(3, \infty)$. f is concave down on (1,3). There are two inflection points, at 1 and at 3. One possible graph of f is in the first graph below.

(d) The critical points are -1 and 2. f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$. f' changes from positive to negative at -1 so there is a maximum at -1. f is concave up on (0,2) and concave down on $(-\infty, 0)$ and on $(2, \infty)$. There are two inflection points, at 0 and at 2. One possible graph of f is in the second graph below.





17. Absolute Extrema. (a) Absolute maximum (4, 449), absolute minimum (2, -63). (b) Absolute maximum (-.34, 25.68), absolute minimum (2.89, -60.42). (c) Absolute maximum (1, 18), absolute minimum (3.2, -21).

18. Optimization Problems.

(a) (i) Increasing (0, 2.5), decreasing $(2.5, \infty)$. (ii) Largest decrease when $C'' = 0 \Rightarrow t = 5$ hours after the drug is administered. $C'(5) \approx -.27\ \mu\text{g/cm}^3$ per hour.

(b) (i) $B''(t) = \frac{10}{9\sqrt[3]{8-3t}} \Rightarrow$ critical point of B' is $8 - 3t = 0 \Rightarrow t = \frac{8}{3} \approx 2.67$. The sign of B'' is changing from negative to positive at $\frac{8}{3}$ so there is a minimum at $\frac{8}{3}$. $B'(\frac{8}{3}) = 0$ kg per month and $B(\frac{8}{3}) = 5$ kg. So, the population is increasing at a decreasing rate in the first 2 and $\frac{2}{3}$ months. At 2 and $\frac{2}{3}$ months, it reaches its lowest rate of 0 kg/month. After that it starts increasing at an increasing rate. (ii) The only critical point $\frac{8}{3}$ is not in the interval [3,6] so it is sufficient to evaluate the function at the endpoints 3 and 6. $B(3) \approx 5.11$, $B(6) \approx 10.16$, and so the maximum is 10.16 at $t = 6$ and the minimum is 5.11 at $t = 3$.

(c) The objective is $F = T^2 + 50P$ and the constraint is $PT = 5000$. The critical points are at $T = 50$ and $T = 0$. There is a minimum at $T = 50$ and no extreme value at $T = 0$. When $T = 50$, $P = 100$ so the pressure of 100 kPa and the temperature of 50° F minimize F.

(d) Using r for the radius of the base and h for the height, $S = 2r^2\pi + 2r\pi h$ is the objective. The constraint is that the volume $r^2\pi h$ is $\frac{1}{2}$. The critical value of the function $S = 2r^2\pi + 2r\pi\frac{1}{2r^2\pi} = 2r^2\pi + \frac{1}{r}$ is $4r^3\pi = 1 \Rightarrow r = \frac{1}{\sqrt[3]{4\pi}} \approx 0.43$ S'' is positive for $r > 0$ and so there is a minimum at 0.43. When $r = 0.43$, $h = 0.86$ so the radius of the base of 4.3 cm and the height of 8.6 cm minimize the amount of the material for the can.

(e) Use the square of the distance $D^2 = (x-2)^2 + (y-4)^2$ as the objective. The constraint is $y^2 = 2x - 2$. $D^2 = (\frac{1}{2}y^2 - 1)^2 + (y-4)^2 = \frac{1}{4}y^4 - 8y + 17 \Rightarrow \frac{d}{dy}D^2 = y^3 - 8$ so the critical point is $y = 2$. The second derivative $3y^2$ is positive at $y = 2$ so that D^2 has a minimum at $y = 2$. When $y = 2, x = 3$. Thus, the point $(3, 2)$ on $y^2 = 2x - 2$ is the closest to (2,4).

(f) If (x, y) is the upper right vertex of the rectangle, the dimensions of the rectangle are $2x$ and y so the area is $A = 2xy$. With the constraint $y = 12 - x^2$, the objective is $A = 2x(12 - x^2) = 24x - 2x^3$. The critical point $x = 2$ produces a maximum since $A''(2) = -24 < 0$. When $x = 2, y = 8$ so the base 4 and height 8 produce the largest area.