Calculus 1
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The Substitution

Many integrals cannot be evaluated using the rules we covered so far (and many others cannot be evaluated even by methods we shall cover in Calculus 2). Still there is a class of integrals which can be evaluated using a method known as the **substitution**. Namely, consider the case when the integrand is a constant multiple of a function of the form

$$f(g(x))\ g'(x)$$

and an antiderivative of $f(x)$ can be found to be $F(x)$. In this case $F'(x) = f(x)$ and the Chain Rule applied to the function $F(g(x))$ produces the function $F'(g(x))g'(x) = f(g(x))g'(x)$. This implies that $F(g(x)))$ is an antiderivative of $f(g(x))g'(x)$.

$$\frac{d}{dx}(F(g(x))) = f(g(x))\ g'(x) \quad \Rightarrow \quad \int f\,(g(x))\ \ g'(x)\ dx = F(g(x)) + c.$$

The last formula is known as the Substitution Rule. You can think of the function $g(x)$ as the inner function and consider the Substitution Rule to be applicable if the integrand is of the form

constant multiple of	$f(g(x))$	$\cdot$	$g'(x)$
	the composite	$\cdot$	the derivative of the inner

To check if the integrand is of this form and perform substitution, use the following steps.

1. Start by analyzing the integrand, determining whether you can evaluate the integral directly, without the substitution, or, if the substitution is appropriate, follow the steps below.
2. **Identify the inner function** $u = g(x)$ (usually the term under a radical, term in parenthesis, denominator, exponent...).
3. **Find the differential** $du = g'(x)dx$. and solve for $dx = \frac{du}{g'(x)}$.
4. **Substitute** the inner function $g(x)$ with u and dx with dx from previous step.
5. The substitution is successful if there are <u>**no terms with x left**</u> in the integrand. You may need to simplify the integrand and use the relation $u = g(x)$ again.
6. If you obtain a simpler integrand than the initial one and can evaluate it using the rules of integration, the substitution method worked.
7. Then **integrate** the integrand.
8. Finally, **substitute back** using that $u = g(x)$ so that your final answer is in terms of x again.

We illustrate this method in the next examples.

Example 1. Determine if the substitution method is appropriate for evaluating the following integrals and, if it is, evaluate them using substitution.

(a) $\displaystyle\int \frac{3}{(4x+1)^2}\,dx$ (b) $\displaystyle\int \frac{x}{\sqrt[3]{x^2+3}}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt[3]{x^2+3}}\,dx$

Solution. (a) Follow the steps of the substitution method.

1. The power rule cannot be applied directly since the integrand $3(4x+1)^{-2}$ is a composite of a power function $3u^{-2}$ and a linear function $4x+1$. Try the substitution.
2. Identify the inner function $u = 4x+1$.
3. Find the differential $du = 4dx$ and solve for $dx = \frac{du}{4}$.
4. Substitute $4x+1$ with u and dx with $\frac{du}{4}$. Obtain the following

$$\int 3(4x+1)^{-2}\,dx = \int 3(u)^{-2}\,\frac{du}{4}$$

5. The substitution is successful since there are no terms with x left.
6. Factor the constants out to simplify the integration. The integrand u^{-2} is now simple and you can integrate it using the Power Rule.
7. Then integrate the integrand.

$$\int 3u^{-2}\,\frac{du}{4} = \frac{3}{4}\int u^{-2}\,du = \frac{3}{4}\,\frac{1}{-2+1}\,u^{-2+1} + c = \frac{-3}{4}\,u^{-1} + c = \frac{-3}{4u} + c.$$

8. Finally, substitute back using that $u = 4x+1$ so that your final answer is in terms of x again.

$$\frac{-3}{4u} + c = \frac{-3}{4(4x+1)} + c.$$

(b)

1. The integral cannot be evaluated directly since the integrand is complex function. Try the substitution.
2. Identify the inner function $u = x^2+3$.
3. Find the differential $du = 2xdx$ and solve for $dx = \frac{du}{2x}$.
4. Substitute x^3+3 with u and dx with $\frac{du}{2x}$. Obtain the following

$$\int x(x^2+3)^{-1/3}\,dx = \int x(u)^{-1/3}\,\frac{du}{2x}$$

5. The substitution is successful since the x terms cancel and we are left with the x-free integrand.

$$\int (u)^{-1/3} \frac{du}{2}$$

6. Factor the constant $\frac{1}{2}$ out to simplify the integration. The integrand $u^{-1/3}$ is now simple and you can integrate it using the Power Rule.

7. Then integrate.

$$\int u^{-1/3} \frac{du}{2} = \frac{1}{2} \int u^{-1/3}\, du = \frac{1}{2} \frac{1}{-\frac{1}{3}+1} u^{-1/3+1} + c = \frac{3}{4} u^{2/3} + c = \frac{3}{4} \sqrt[3]{u^2} + c.$$

8. Finally, substitute back using that $u = x^2 + 3$ so that your final answer is in terms of x again.

$$\frac{3}{4} \sqrt[3]{u^2} + c = \frac{3}{4} \sqrt[3]{(x^2+3)^2} + c.$$

(c)

1. The integral cannot be evaluated directly since the integrand is complex function. Try the substitution.

2. Identify the inner function $u = x^2 + 3$.

3. Find the differential $du = 2xdx$ and solve for $dx = \frac{du}{2x}$.

4. Substitute $x^3 + 3$ with u and dx with $\frac{du}{2x}$. Obtain the following

$$\int (x^2+3)^{-1/3}\, dx = \int (u)^{-1/3} \frac{du}{2x}$$

5. The substitution is **not** successful since there is x terms left in the integrand and representing x as $x = \pm\sqrt{u-3}$ will make the integral more complex.

   Trying to change u from x^2+3 to x^2, $(x^2+3)^{1/3}$, or $(x^2+3)^{-1/3}$ also leads to similar problems. Conclude that the substitution is not appropriate method for this integral.

A bit more delicate situation arises in the case as in the example below. In these integrals, after picking the inner function for u and making the substitution, the integrand needs to be simplified further in order to complete the transition from x to u. You will encounter these integrals again in Calculus 2 when finding the surface area of revolving parametric curve.

Example 2. Evaluate the following integrals.

(a) $\int x\sqrt{x-3}\, dx$ (b) $\int x^3\sqrt{1+x^2}\, dx$

Solution. (a) Start with $u = x - 3$ and differentiating to get $du = dx$. Substitute and obtain $\int x\sqrt{u}\, du$. Use the formula $u = x - 3$ to express x in terms of u as $x = u + 3$. The integral becomes $\int (u+3)\sqrt{u}du = \int (u^{3/2} + 3u^{1/2})du = \frac{2}{5}u^{5/2} + 3\frac{2}{3}u^{3/2} + c = \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + c.$

(b) Start with picking the inner function $1+x^2$ for u. $u = x^2+1 \Rightarrow du = 2xdx \Rightarrow dx = \frac{du}{2x}$. Substitute and obtain $\int x^3\sqrt{u}\,\frac{du}{2x} = \frac{1}{2}\int x^2\sqrt{u}\,du$. Use the formula $u = x^2+1$ to express x^2 in terms of u as $x^2 = u-1$ and substitute that in the integral as well. The integral becomes $\frac{1}{2}\int(u-1)\sqrt{u}du = \frac{1}{2}\int(u^{3/2}-u^{1/2})du = \frac{1}{2}(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2})+c = \frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+c$.

Practice Problems.

1. Determine if the substitution method is appropriate for evaluating the following integrals and, if it is, evaluate them using substitution.

   (a) $\displaystyle\int(3x{+}5)^6\;dx$ (b) $\displaystyle\int(2x{+}1)^3\,dx$ (c) $\displaystyle\int\frac{x}{(x^2+3)^2}\,dx$

   (d) $\displaystyle\int\frac{(x^2+1)^2}{x^2}\,dx$ (e) $\displaystyle\int\frac{x}{(2x+1)^2}\,dx$ (f) $\displaystyle\int\frac{x^2}{\sqrt{x^3-5}}\,dx$

   (g) $\displaystyle\int\frac{1}{\sqrt{x^4+7}}\,dx$ (h) $\displaystyle\int\frac{6}{\sqrt[3]{3x+5}}\,dx$ (i) $\displaystyle\int 3x\left((x^2+1)^2-\sqrt{x^2+1}\right)dx$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

   (a) $f'(x) = \sqrt{2x+9}$ and $f(0)=5$
   (b) $f'(x) = \frac{10}{\sqrt{4x+1}}$ and $f(0)=3$
   (c) $f'(x) = x^2\sqrt{4x^4+9x^2}$ and $f(0)=-2$.

3. Suppose that the velocity of an object is given by the function

$$v(t) = \frac{t}{\sqrt{t^2+9}}$$

   where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t=4$ seconds, the position function $s(t)=8$ feet, determine the position function $s(t)$.

Solutions.

1. (a) Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int(3x+5)^6\;dx = \int u^6\frac{du}{3} = \frac{1}{3}\int u^6du = \frac{1}{3}\frac{u^7}{7}+c = \frac{u^7}{21}+c = \frac{(3x+5)^7}{21}+c$.
   (b) Use substitution $u = 2x+1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int(2x+1)^3\;dx = \int u^3\frac{du}{2} = \frac{1}{2}\int u^3du = \frac{1}{2}\frac{u^4}{4}+c = \frac{u^4}{8}+c = \frac{(2x+1)^4}{8}+c$.
   (c) Use substitution $u = x^2+3$. Then $du = 2xdx$ so $dx = \frac{du}{2x}$. $\int\frac{x}{(x^2+3)^2}\;dx = \int\frac{x}{u^2}\frac{du}{2x} = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\frac{1}{-1}u^{-1}+c = \frac{-1}{2u}+c = \frac{-1}{2(x^2+3)}+c$.
   (d) Note that substitution $u = x^2+1$ does not work since there is a remaining term x in denominator after substituting. The integral can be evaluated without substitution since the

integrand simplifies as $\frac{(x^2+1)^2}{x^2} = \frac{x^4+2x^2+1}{x^2} = x^2+2+x^{-2}$ so that the integral becomes $\int(x^2+2+x^{-2})dx = \frac{1}{3}x^3+2x+\frac{1}{-1}x^{-1}+c = \frac{x^3}{3}+2x-\frac{1}{x}+c$.

(e) Use substitution $u = 2x+1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. Substitute and obtain $\int x(2x+1)^{-2}dx = \int xu^{-2}\frac{du}{2}$. Use the formula $u = 2x+1$ to express x in terms of u as $x = \frac{1}{2}(u-1)$. The integral becomes $\int \frac{1}{2}(u-1)u^{-2}\frac{du}{2} = \frac{1}{4}\int(u^{-1}-u^{-2})du = \frac{1}{4}(\ln|u| - \frac{1}{-1}u^{-1})+c = \frac{1}{4}\ln|u|+\frac{1}{4u}+c = \frac{1}{4}\ln|2x+1|+\frac{1}{4(2x+1)}+c$.

(f) Use substitution $u = x^3-5$. Then $du = 3x^2dx$ so $dx = \frac{du}{3x^2}$. $\int \frac{x^2}{\sqrt{x^3-5}}\ dx = \int \frac{x^2}{\sqrt{u}}\frac{du}{3x^2} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\frac{u^{1/2}}{1/2}+c = \frac{2\sqrt{u}}{3}+c = \frac{2\sqrt{x^3-5}}{3}+c$ or $\frac{2}{3}\sqrt{x^3-5}+c$.

(g) The integral cannot be evaluated by substitution $u = x^4+7$. Substitutions $u = \sqrt{x^4+7}$ or $u = (x^4+7)^{-1/2}$ are equally ineffective.

(h) Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int \frac{6}{\sqrt[3]{3x+5}}\ dx = 6\int \frac{1}{u^{1/3}}\frac{du}{3} = \frac{6}{3}\int u^{-1/3}du = 2\,\frac{u^{2/3}}{2/3}+c = 3u^{2/3}+c = 3(3x+5)^{2/3}+c$ or $3\sqrt[3]{(3x+5)^2}+c$.

(i) Use substitution $u = x^2+1$ so that $du = 2xdx$ and $dx = \frac{du}{2x}$ to obtain $\int 3x(u^2-u^{1/2})\ \frac{du}{2x} = \frac{3}{2}\int(u^2-u^{1/2})\ du = \frac{3}{2}(\frac{1}{3}u^3-\frac{2}{3}u^{3/2})+c = \frac{1}{2}u^3-u^{3/2}+c = \frac{1}{2}(x^2+1)^3-(x^2+1)^{3/2}+c$.

2. (a) $f(x) = \int f'(x)\ dx = \int \sqrt{2x+9}\,dx$. Use substitution $u = 2x+9$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int \sqrt{2x+9}\ dx = \int u^{1/2}\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2}+c = \frac{1}{3}u^{3/2}+c = \frac{1}{3}(2x+9)^{3/2}+c$. Using $f(0) = 5$ to solve for c, you have that $5 = \frac{1}{3}\sqrt{9^3}+c = \frac{27}{3}+c \Rightarrow 5 = 9+c \Rightarrow c = -4$. Thus, $f(x) = \frac{1}{3}(2x+9)^{3/2}-4$.

(b) $f(x) = \int f'(x)\ dx = \int \frac{10}{\sqrt{4x+1}}\,dx$. Use substitution $u = 4x+1$. Then $du = 4dx$ so $dx = \frac{du}{4}$. $\int \frac{10}{\sqrt{4x+1}}\ dx = 10\int u^{-1/2}\frac{du}{4} = \frac{10}{4}\frac{u^{1/2}}{1/2}+c = 5u^{1/2}+c = 5\sqrt{4x+1}+c$. Using $f(0) = 3$ to solve for c, you have that $3 = 5\sqrt{0+1}+c = 5+c \Rightarrow 3 = 5+c \Rightarrow c = -2$. Thus, $f(x) = 5\sqrt{4x+1}-2$.

(c) $f(x) = \int x^2\sqrt{4x^4+9x^2}dx$. First simplify the integrand as $\int x^2\sqrt{x^2(4x^2+9)}dx = \int x^2\,x\,\sqrt{4x^2+9}dx = \int x^3\sqrt{4x^2+9}dx$. Use the substitution $u = 4x^2+9 \Rightarrow du = 8xdx \Rightarrow dx = \frac{dx}{8x}$. Substitute and obtain $\int x^3\sqrt{u}\,\frac{du}{8x} = \frac{1}{8}\int x^2\sqrt{u}\,du$. Use the formula $u = 4x^2+9$ to express x^2 in terms of u as $x^2 = \frac{1}{4}(u-9)$ and substitute that in the integral as well. The integral becomes $\frac{1}{32}\int(u-9)\sqrt{u}du = \frac{1}{32}\int(u^{3/2}-9u^{1/2})du = \frac{1}{32}(\frac{2}{5}u^{5/2}-9\frac{2}{3}u^{3/2})+c = \frac{1}{80}(4x^2+9)^{5/2}-\frac{3}{16}(4x^2+9)^{3/2}+c$.

Since $f(0) = -2$, $-2 = \frac{1}{80}9^{5/2}-\frac{3}{16}9^{3/2}+c = \frac{243}{80}-\frac{81}{16}+c \Rightarrow -2 = \frac{-81}{40}+c \Rightarrow c = \frac{1}{40}$ resulting in $f(x) = \frac{1}{80}(4x^2+9)^{5/2}-\frac{3}{16}(4x^2+9)^{3/2}+\frac{1}{40}$.

3. $s(t) = \int v(t)\ dt = \int \frac{t}{\sqrt{t^2+9}}\,dt$. Use substitution $u = t^2+9$. Then $du = 2tdt$ so $dt = \frac{du}{2t}$. $\int \frac{t}{\sqrt{t^2+9}}\ dt = \int \frac{t}{u^{1/2}}\frac{du}{2t} = \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\frac{u^{1/2}}{1/2}+c = u^{1/2}+c = \sqrt{t^2+9}+c$. Using $s(4) = 8$ to solve for c, you have that $8 = \sqrt{4+9}+c = 5+c \Rightarrow 8 = 5+c \Rightarrow c = 3$. Thus, $s(t) = \sqrt{t^2+9}+3$.