Calculus 2
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Autonomous Differential Equations and Population Dynamic

If a differential equation is of the form

$$y' = f(y),$$

it is called **autonomous**. Note that an autonomous equation is a separable differential equation.

If f is zero at a, then the horizontal line $y = a$ is a solution. This solution is called the **equilibrium solution** and a is called a critical point. After finding the equilibrium solutions, check the sign of f. On intervals of y with $y' = f(y)$ positive, the solutions y are increasing and on intervals of y with $y' = f(y)$ negative, the solutions y are decreasing. Thus, the analysis of the sign of $f(y)$ can tell us a lot about the graph of the solutions.

If the solutions asymptotically approach the equilibrium solution for $t \to \infty$, regardless of the values of initial conditions, then the equilibrium solution is called **asymptotically stable** solution. If the solutions are not converging towards the equilibrium solution for all values of initial conditions, then the equilibrium solution is called **unstable** solution. In case that on one side the solutions are converging towards the equilibrium solution and on the other they do not, we say that the equilibrium solution is **semistable**.

In some cases, it may be difficult to obtain an explicit formula of the solution. In those cases, getting a graph of an autonomous equation may provide valuable information about the solution.

Applications - Models of population growth

The simplest model of population growth is one describing the population that the population changes at a rate proportional to its size. The differential equation model for that situation is

$$\frac{dP}{dt} = kP$$

We encounter this situation in examples when the percent birth rate is b and the percent death rate is c so that $\frac{dP}{dt} = bP - cP$. Let $k = b - c$. If $b > c$ (so $k > 0$) the population will be increasing, if $b < c$ (so $k < 0$) the population will be decreasing and if $b = c$ (so $k = 0$) the population will remain constant.

If the initial size is P_0, separating the variables you obtain $\frac{dP}{P} = kdt \Rightarrow \ln P = kt + c \Rightarrow P = e^{kt+c} = Ce^{kt}$. Using the initial condition, you obtain that $C = P_0$. So, the solution is $P = P_0e^{kt}$.

Limited Growth. If a given environment has limited resources to support the population growth, the population might not increase indefinitely. Suppose that the rate of increase is proportional to the population size and the difference between a constant K (called **carrying capacity**) limiting the growth. The differential equation model for that situation is

$$\frac{dP}{dt} = kP(K - P)$$

Graphing the solution of this autonomous equation, we can see that the population size increases to P if the initial size P_0 is smaller than K but the growth does not increase the capacity K. If $P_0 > K$, the population size decreases to K as the population is too large to grow due to the limiting resources of the environment. If $P_0 = K$, the solution is constant $P = K$.

Note that in this case, the equilibrium solution $P = K$ is stable since $\lim_{t\to\infty} P = K$ regardless of the initial size.

A model with a threshold level. Suppose that a given population can keep increasing just if the initial size is large enough and it dies out otherwise. The level that allows the increase of population is called a **threshold level**. In this case the rate of increase is proportional to the population size and the difference of the present population size and the threshold level T. The differential equation model for that situation is

$$\frac{dP}{dt} = kP(P - T)$$

Graphing the solution of this autonomous equation, we can see that the population size decreases to 0 if the initial size P_0 is smaller than T. If $P_0 > T$, the population size increases without a bound. If $P_0 = T$, the solution is constant $P = T$. Note that in this case, the equilibrium solution $P = T$ is unstable.

Practice Problems.

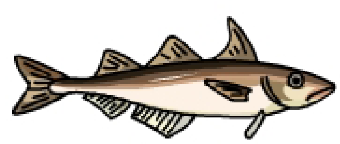
1. Sketch the graph of solutions of the following equations.

   a) $y' = y^2 - 2y$ b) $y' = -y^2 + 2y$ c) $y' = (y + 1)(y - 2)^2$
   d) $y' = y(y + 1)(y - 2)$ e) $y' = y(2 - y)^2(5 - y)^3$

2. The size of a population of rabbits is modeled by differential equation $P' = -kP(100 - P)$ where k is a positive parameter.

   a) Estimate the number of rabbits after a long period of time if the initial size of the population is 103 rabbits.
   b) Estimate the number of rabbits after long period of time if the initial size of the population is 99 rabbits.
   c) If $k = 0.02$ per year, use the Euler program to estimate the size of the population after 4 years if the population was 99 initially. Use 0.5 for the step size.

3. The Pacific halibut fishery is modeled by differential equation $B' = kB(K - B)$ where B is the biomass (total mass of the members of the population) in kilograms at time t, $K = 8 \cdot 10^7$ kg and $k = 8.7 \cdot 10^{-9}$ per year.



a) Estimate the biomass after many years if the initial biomass is $3 \cdot 10^6$.

b) Estimate the biomass after many years if the initial biomass is $9 \cdot 10^7$.

c) If the biomass is $2 \cdot 10^7$ kg initially, use the Euler program to estimate the biomass 5 years later. Use 0.5 for the step size.

4. In a seasonal-growth model for population growth, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food. The rate of change of the population is proportional to the size of population multiplied with a periodic function. Thus, this situation can be modeled by the differential equation

$$\frac{dP}{dt} = kP\cos(rt - \phi) \quad P(\frac{\phi}{r}) = P_0,$$

where k, r and P_0 are positive constants and ϕ a non-negative constant. Find the solution of this differential equation.

Solutions.

1. a) To find equilibrium solution solve $y^2 - 2y = y(y-2) = 0 \Rightarrow y = 0$ and $y = 2$. Then analyze the sign of y'. $+ \;\; 0 \;\; - \;\; 2 \;\; +$. Use this information to sketch the graph of general solutions: above $y = 2$, and below $y = 0$, the solutions are increasing, and between $y = 0$ and $y = 2$ the solutions are decreasing. From the graph, you can see that $y = 0$ is asymptotically stable and $y = 2$ is unstable.

b) $-y^2 + 2y = -y(y-2) = 0 \Rightarrow y = 0$ and $y = 2$. $- \;\; 0 \;\; + \;\; 2 \;\; -$. Thus, above $y = 2$, and below $y = 0$, the solutions are decreasing, and between $y = 0$ and $y = 2$ the solutions are increasing. Conclude that $y = 2$ is asymptotically stable and $y = 0$ is unstable.

c) Equilibrium solutions: $y = -1$ and $y = 2$. Sign of y' : $- \;\; {-1} \;\; + \;\; 2 \;\; +$. Thus, below $y = -1$, the solutions are decreasing. Between $y = -1$ and $y = 2$ and above $y = 2$ the solutions are increasing. Conclude that $y = 2$ is semistable and $y = -1$ is unstable.

d) Equilibrium solutions: $y = -1$, $y = 0$ and $y = 2$. Sign of y' : $- \;\; {-1} \;\; + \;\; 0 \;\; - \;\; 2 \;\; +$.
Conclude that $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable.

e) $y' = y(2-y)^2(5-y)^3 = 0 \Rightarrow y = 0$, $(2-y)^2 = 0$ or $(5-y)^3 = 0 \Rightarrow y = 0$, $2 - y = 0$ or $5 - y = 0$. So, the equilibrium solutions are $y = 0$, $y = 2$ and $y = 5$. Sign of y' : $- \;\; 0 \;\; + \;\; 2 \;\; + \;\; 5 \;\; -$.
Conclude that $y = 0$ is unstable, $y = 2$ is semistable and $y = 5$ is stable.

2. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solutions. $-kP(100 - P) = 0 \Rightarrow P = 0$ and $P = 100$. Sign of P' : $+ \;\; 0 \;\; - \;\; 100 \;\; +$. Thus, with initial condition above $P = 100$ (and below $P = 0$ but that is not relevant in this case)

the solutions are increasing. In particular, if the initial population size is 103, the population will be increasing. Thus $\lim_{t\to\infty} P = \infty$. So, the population size increases without bounds. The solutions with initial conditions between $P = 0$ and $P = 100$ are decreasing. In particular, if the initial population size is 99, the population will be decreasing to 0. Thus $\lim_{t\to\infty} P = 0$. So, the population size decreases to 0 in this case.

c) Enter the equation $y' = -0.02y(100 - y)$, 0 for x-initial, 99 for y-initial, 4 for x-final and 0.5 for the step size and obtain that the population size decreased to about 7.63 (can round to 8) four years after.

3. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solutions. $B' = kB(8 \cdot 10^7 - B) = 0 \Rightarrow B = 0$ and $B = 8 \cdot 10^7$. Sign of B' : $\overset{-}{}\underset{0}{|}\overset{+}{}\underset{8\cdot 10^7}{|}\overset{-}{}$.
Thus, with initial condition above $B = 8 \cdot 10^7$ (and below $B = 0$ but that is not relevant in this case) the solutions are decreasing. In particular, if the initial biomass is $9 \cdot 10^7$, the biomass will be decreasing to $8 \cdot 10^7$ so $\lim_{t\to\infty} B = 8 \cdot 10^7$. The solutions with initial conditions between $B = 0$ and $B = 8 \cdot 10^7$ are increasing. In particular, if the initial biomass is $3 \cdot 10^7$, the biomass will be increasing to $8 \cdot 10^7$. Thus $\lim_{t\to\infty} B = 8 \cdot 10^7$ as well.

   c) Enter the equation $y' = 8.7 \cdot 10^{-9} y(8 \cdot 10^7 - y)$, 0 for x-initial, $2 \cdot 10^7$ for y-initial, 5 for x-final and 0.5 for the step size and obtain that the biomass increased to $74242300 \approx 7.4 \cdot 10^7$ kg in 5 years.

4. Separate the variables $\frac{dP}{dt} = kP\cos(rt-\phi) \Rightarrow \frac{dP}{P} = k\cos(rt-\phi)dt \Rightarrow \ln P = \int k\cos(rt-\phi)dt \Rightarrow \ln P = \frac{k}{r}\sin(rt-\phi) + c \Rightarrow P = e^{\frac{k}{r}\sin(rt-\phi)+c} = Ce^{\frac{k}{r}\sin(rt-\phi)}$. Using the initial condition, obtain that $C = P_0$. Thus $P = P_0 e^{\frac{k}{r}\sin(rt-\phi)}$.