# Definite Integral. The Left and Right Sums

The definite integral arises from the question of finding the area between a given curve and x-axis on an interval and it can be used to find the total change in some quantity knowing the rate of change of that quantity.

The area under a curve can be easily calculated if the curve is given by a simple formula. For example, if a function is a positive constant $f(x) = c$, the area under this curve on interval $[a, b]$ is the area of the rectangle with sides $b - a$ and c. Thus the area is $A = (b - a)c$.
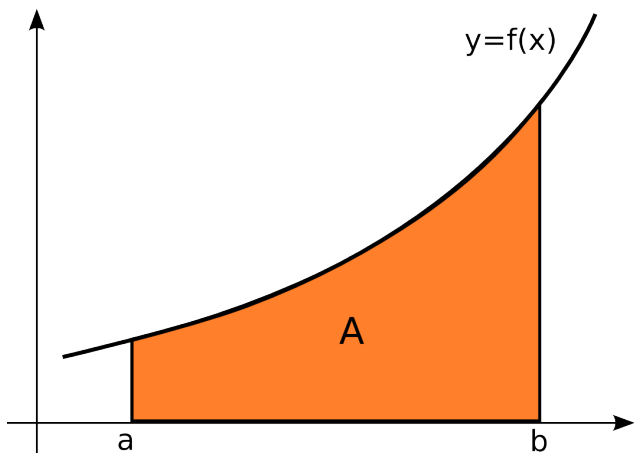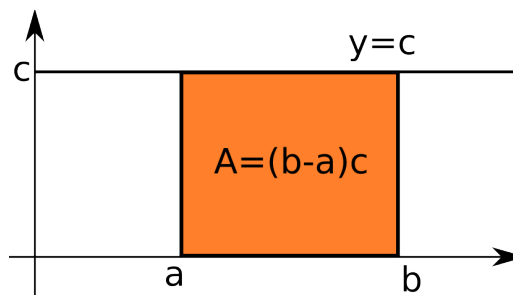
We are interested in calculating the area A under the curve f for $a \leq x \leq b$ for **any** non-negative, continuous curve $f(x)$. This, in general, rather complex problem can be approached the following idea: we divide the interval $[a, b]$ in small subintervals and approximate the area under $f(x)$ on each subinterval with the area of a rectangle with height $f(x_i)$ where x_i is a point in i-th subinterval of $[a, b]$.

Let us make this idea more precise using the sums of rectangles known as **the left and the right sums**.

Approximating the area.

Step 1. The partition. Divide $[a, b]$ into n-pieces. Get the points $a = x_0, x_1, \ldots, x_{n-1}$, $x_n = b$. This is called the **partition** of $[a, b]$. The distance between each two points is $h = \frac{b-a}{n}$.

Step 2. The rectangles. The left and right sums differ in the way how we choose the point from the subinterval which will produce the height of the rectangle.
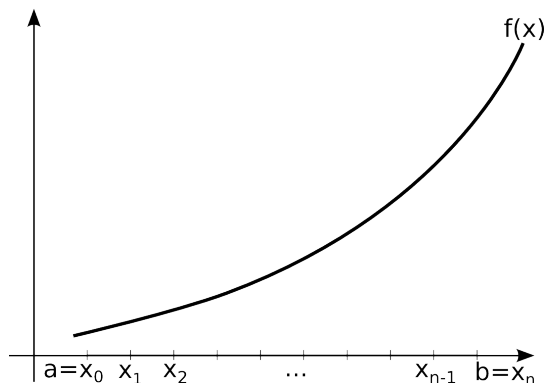
To approximate the area using the left sum, choose the **left** endpoint of each subinterval and evaluate the function at it to obtain the height of the rectangle. For example, x_0 is the left endpoint of the first subinterval $[x_0, x_1]$ and so the height of the first rectangle is $f(x_0)$.

The area of the first rectangle is $A_0 = hf(x_0)$. The left endpoint of the second interval $[x_1, x_2]$ is x_1 and the second height is $f(x_1)$. The area of the second rectangle is $A_1 = hf(x_1)$. Continuing this process we obtain that the area of the $i+1$-st rectangle is $A_i = hf(x_i)$.

The process ends with the n-th rectangle of the area $A_{n-1} = hf(x_{n-1})$. The sum of all the rectangles is
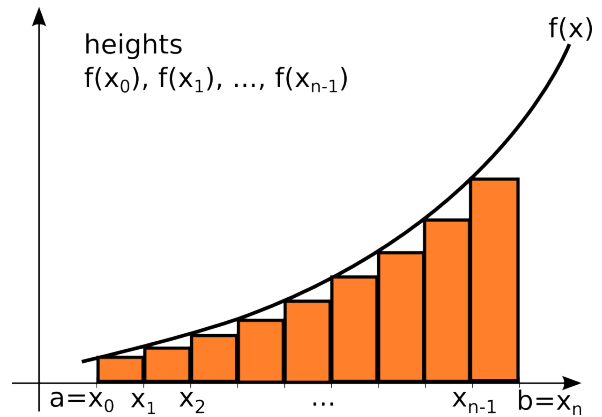
$$
\begin{aligned}
\text{Left sum} &= hf(x_0) + hf(x_1) + \ldots + hf(x_{n-1}) = h\ (f(x_0) + f(x_1) + \ldots + f(x_{n-1})) \\
&= \tfrac{b-a}{n}\ (f(x_0) + f(x_1) + \ldots + f(x_{n-1}))\,.
\end{aligned}
$$

Analogously, to approximate the area using the right sum, choose the **right** endpoint of each subinterval to form a rectangle. The point x_1 is the right endpoint of the first subinterval $[x_0, x_1]$ and so the height of the first rectangle is $f(x_1)$ and the area is $A_1 = hf(x_1)$. The right endpoint of the second interval $[x_1, x_2]$ is x_2, the second height is $f(x_2)$, and the area of the second rectangle is $A_2 = hf(x_2)$. Continuing this process we obtain that the area of the i-th rectangle is $A_i = hf(x_i)$.
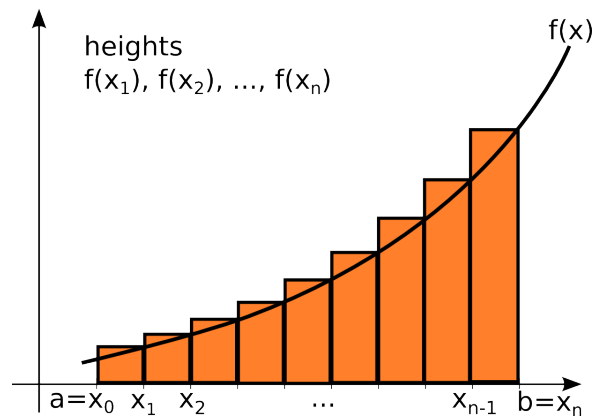
The process ends with the n-th rectangle of the area $A_n = hf(x_n)$. The sum of all the rectangles is

$$
\begin{aligned}
\text{Right sum} &= hf(x_1) + hf(x_2) + \ldots + hf(x_n) = h\ (f(x_1) + f(x_2) + \ldots + f(x_n)) \\
&= \tfrac{b-a}{n}\ (f(x_1) + f(x_2) + \ldots + f(x_n))\,.
\end{aligned}
$$

Thus, the area under a curve can be approximated using both left and right sums.

$$
\text{Area } A \approx \text{ Left sum } = \frac{b-a}{n}\ (f(x_0) + f(x_1) + \ldots + f(x_{n-1}))
$$

$$
\text{Area } A \approx \text{ Right sum } = \frac{b-a}{n}\ (f(x_1) + f(x_2) + \ldots + f(x_n))
$$

Example 1. Approximate the are under $f(x) = x^2$ on [0,2] using the left and right sums with 2 subintervals.

Solution. Note that $a = 0, b = 2$ and $n = 2$. Thus $\frac{b-a}{n} = \frac{2-0}{2} = 1$. The partition consists of 3 points $a = x_0 = 0$, $x_1 = 1$ and $b = x_2 = 2$.

Compute the corresponding y-values to be $f(0) = 0^2 = 0$, $f(1) = 1^1 = 1$ and $f(2) = 2^2 = 4$. Thus the left sum is $L = \frac{2-0}{2}(0 + 1) = 1$ and the right sum is $L = \frac{2-0}{2}(1 + 4) = 5$.


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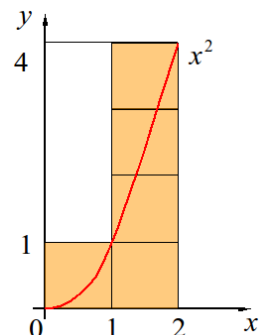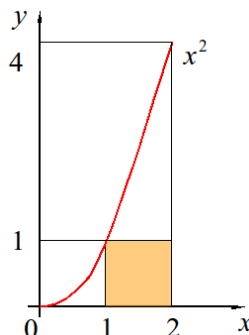
From the graph, we can conclude that the left sum 1 is an underestimate and the right sum 5 is an overestimate. Thus $1 \leq A \leq 5$.

Note that the approximation is not very accurate since we do not even have the first digit of the exact answer correct. Increasing the number of subintervals, that is using larger values of n, increase the accuracy of approximation as the next figure illustrates.
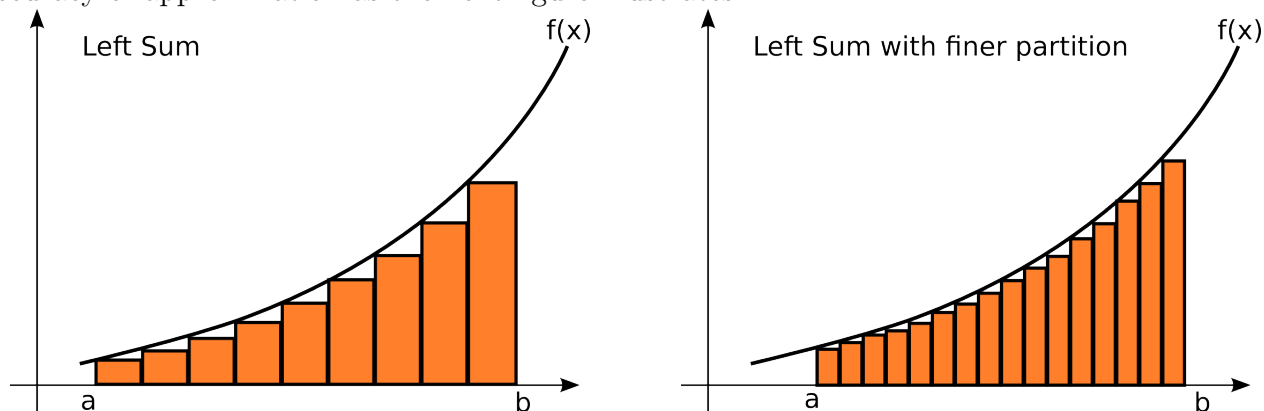
This fact suggests that finding the exact area under the curve can be obtained by taking the limit of any of these sums when the number of partitions n increases to infinity.
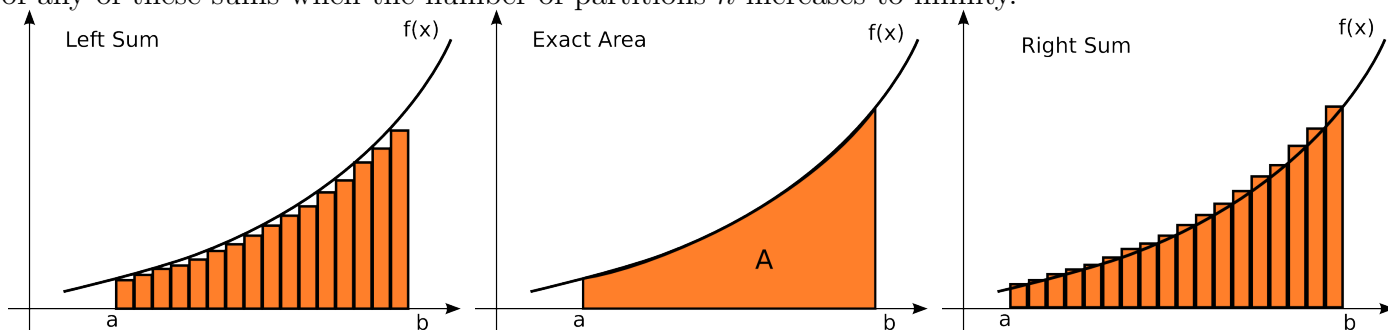
If f is increasing and nonnegative continuous function as on the last several figures, the exact area will be sandwiched between the left sum from below and the right sum from above (thus left and right sums are "two pieces of bread"). The limit of both left and right sum when $n \to \infty$ approaches the same value which is the exact value of the area under the curve ("ham") .

Area= ham, left and right sums = bread

If f is an *increasing*, nonnegative, continuous function, the left sum always produces an underestimate and the right sum an overestimate. For decreasing, nonnegative, continuous functions this is the opposite. For the functions with extreme values neither has to be the case.

When f is a nonnegative, continuous function, not necessary increasing, the area can still be sandwiched between the *lower sum* (obtained by forming rectangles using points with the smallest height on each subinterval) and the *upper sum* (obtained by forming rectangles using points with the highest height on each subinterval). Using the Closed Interval Method, such points can be found.

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closed interval.

$$\boxed{\text{Lower Sum} \leq \text{Area } A \leq \text{Upper Sum}}$$

This also illustrates that the limit of the sum of rectangles formed by **any** point $\overline{x_i}$ from the i-th subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$, exists. Moreover this limit exists for any continuous function (not necessarily nonnegative) defined on a
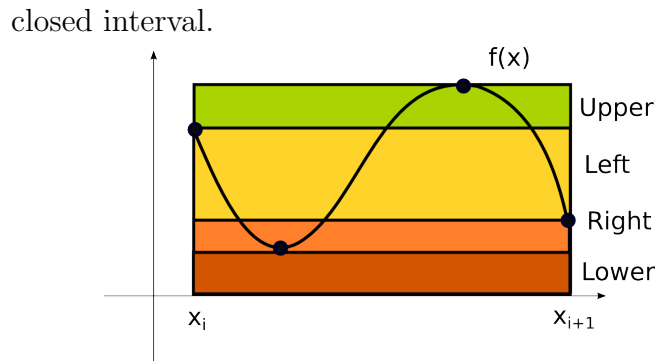
The sum of the rectangles is $(f(\overline{x_1}) + f(\overline{x_2}) + \ldots + f(\overline{x_n}))h$ and can be written shorter as $\sum_{i=1}^{n} f(\overline{x_i})h$. The length of the subinterval $h = \frac{b-a}{n}$ is also denoted as Δx. The limit $\lim_{n \to \infty} \sum_{i=1}^{n} f(\overline{x_i})\Delta x$ is defined to be the **definite integral** of a continuous function $f(x)$ on an interval $[a, b]$ and it is denoted as $\int_a^b f(x)dx$. Thus we have that

$$\textbf{The Definite Integral} \qquad \int_a^b f(x)\ dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(\overline{x_i})\ \Delta x.$$

The following informal reasoning is the base for the notation $\int_a^b f(x)dx$: the small values of Δx correspond to dx, when the number of subintervals is very large and their number very small, picking values $\overline{x_i}$ from each subinterval amounts to picking all points x from the interval $[a, b]$. In the limiting case, the sum becomes the integral symbol indicating that we pick values x on continuous way from entire interval $[a, b]$.

$$\Delta x \rightsquigarrow dx, \quad f(\overline{x_i}) \rightsquigarrow f(x), \quad \sum_{i=1}^{n} \rightsquigarrow \int_a^b \quad \text{so that} \quad \sum_{i=1}^{n} f(\overline{x_i})\ \Delta x \to \int_a^b f(x)\ dx.$$

Thus, the definite integral provides a way of calculating the area under the curve.
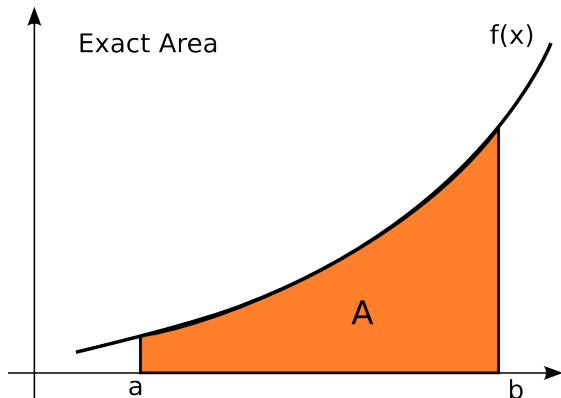
If f nonnegative, continuous function on $[a, b]$, the area A under the curve f above the x-axis, for $a \leq x \leq b$ is

$$\boxed{A = \int_a^b f(x)\ dx.}$$

Before addressing the area under curves which are not nonnegative in the next section, we consider some applications of the left and right sums.

Example 2. A skydiver drops from an airplane. At the end of the first six seconds the diver's speed (in meters per second) is checked, and it reads as follows.

| speed	8	14	19	23	25	26

You can assume that the diver's initial speed is zero. Use the appropriate left and right sums to approximate the distance the diver falls during the six-second period.

Solution. With the restrictions of the measurements above, we approximate the speed during each second with a constant. Thus the distance traveled in each second can be approximated with *the product of the speed and the time.* If the speed values are represented as dependent variable values and time values as the independent variable values, the product of velocity and time corresponds exactly to area of rectangles. Since the speed values are strictly increasing and positive, the left sum provides an underestimate and the right sum the overestimate of the total distance traveled.

Adding the time values to the given table, we obtain the following.

time	0	1	2	3	4	5	6
speed	0	8	14	19	23	25	26

The speed is measured on the time interval $[0,6]$ so we can denote $a = 0$ and $b = 6$. The number of subinterval is $n = 6$. Note that n represents *the number of subintervals not the number of points* (7 points determines 6 subintervals). The left sum is $L = \frac{6-0}{6}(0+8+14+19+23+25) = 89$ and the right sum is $R = \frac{6-0}{6}(8+14+19+23+25+26) = 115$.
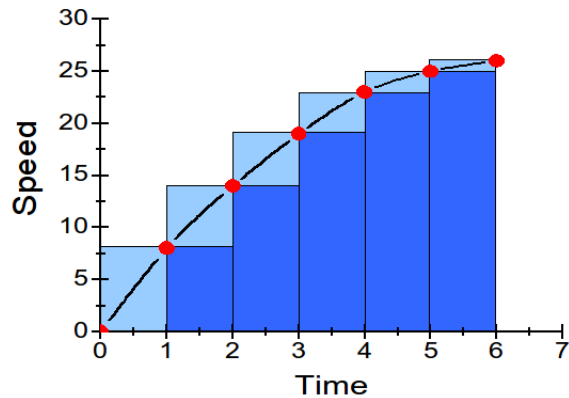
Thus the diver fell by an amount between 89 and 115 meters (note that the units of the answer are meters since the term $\frac{6-0}{6}$ is in seconds and the sums in parenthesis are in meters per seconds).

Example 3. Approximate the area of the lake using the shown measurements of its width which were taken 50 feet apart. | 0 | 88 | 110 | 145 | 180 | 138 | 129 | 93 | 84 | 0 |

Solution. You can consider the measurement values as the dependent variable values at the values of the independent variable starting at 0 and being 50 feet apart.

x	0	50	100	150	200	250	300	350	400	450
y	0	88	110	145	180	138	129	93	84	0
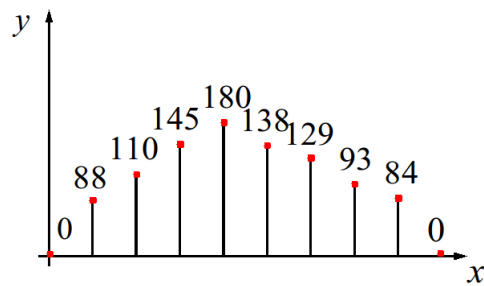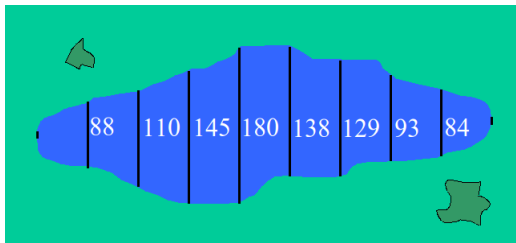
From the table above we can see that $a = 0$ and $b = 50 \cdot 9 = 450$. There are 10 points, thus $n = 9$ subintervals. Alternatively, the measurements are taken 50 feet apart, thus $h = \frac{b-a}{n} = 50$.

The left sum is $L = \frac{450-0}{9}(0+88+110+145+180+138+129+93+84) = 48350$ and the right sum is $R = \frac{450-0}{9}(88+110+145+180+138+129+93+84+0) = 48350$. So, the area of the lake is estimated to be 48350 square feet. Try to generalize the conclusion of this problem: the left and the right sums will be equal whenever $f(x_0) = f(x_n)$.

Left and Right Sum Program for TI83–84

The program below calculates left and right sum of a given function f on a given interval $[a,b]$ with the given number of subintervals n. When executing this program in order to approximate $\int_a^b f(x)dx$ using left and right sums, the function f should be entered as $\mathbf{Y}_1$ and a, b and n should be entered when executing the program.

When entering the program in the calculator, editing or executing and existing program, start by using **PRGM** key.

```
PROGRAM: LFTRGT
Disp ''LOWER BOUND''            (to display Disp, choose PRGM then I/0 menu)
Input A                         (to display Input, choose PRGM then I/0 menu)
Disp ''UPPER BOUND''
Input B
Disp ''NUMBER OF SUBINTERVALS''
Input N
(B-A)/N → D
0 → L
For (I, 0, N-1)                 (to display For, choose PRGM then CTL menu)
A+D*I → X
L+D*Y₁ → L                      (to display Y₁, choose VARS, Y-VARS, then 1: Function)
End                             (to display End, choose PRGM then CTL menu)
Disp ''LEFT SUM'', L
0 → R
For (J, 1, N)
A+D*J → X
R+D* Y₁ → R
End
Disp ''RIGHT SUM'', R
```

You can exit the program editing by using **QUIT**.

Example 4. Use the calculator program to find the left and right sums of $f(x) = x^2$ on interval $[0,2]$ with 50 subintervals.

Solution. Start by entering x^2 as your function $\mathbf{Y}_1$. Then execute the program by **PRGM** and choosing **LFTRGT**. Enter the bounds 0 and 2 and the number of subintervals 50. You should get 2.5872 for the left and 2.7472 for the right sum.

Example 5. Use the Left and Right sums to estimate the area under the curve $f(x) = x^2$ on interval $[0,2]$ to the first two nonzero digits (i.e. make your estimates have the first two nonzero digits equal).

Solution. Since x^2 is positive and increasing on $[0,2]$, the left sum is an underestimate and the right sum is an overestimate of the exact area $\int_0^2 x^2 dx$.

We are given function and the interval but we are not given the number of subintervals. Thus, you can start by entering the given function, the endpoints of the interval and a relatively large n (e.g. 100 or 200) to monitor the left and right sum values L and R. Keep increasing n if necessary until L and R both round to the same value with desired number of significant digits. In that case, you obtain the value of the definite integral you are asked to find.

Note that you do not want to start with n that is too small (e.g. below 10 because it will be most likely that you will have to increase it anyway) nor n that is too large (e.g. 100,000 because it may take a long time for your calculator to produce the answers). Note also that you are not asked to find the smallest possible value of n but *any* value of n that will produce the desired accuracy.

In the previous example we obtained that with $n = 50$ the left sum rounds to 2.6 and the right sum to 2.7. Since we need to have the first two nonzero digits in the left and right sums equal, we need to increase the accuracy.

With $n = 200$, we obtain $2.6467 \approx 2.6$ for the left and $2.6867 \approx 2.7$ for the right which is still not accurate enough. With $n = 300$, we obtain $2.6533 \approx 2.7$ for the left and $2.6800 \approx 2.7$ Thus, we conclude that the area is 2.7 up to the first two nonzero digits.

Practice Problems.

1. Approximate the area under the following curves on the given intervals using left and right sums using the partition with n subintervals.

   (a) $f(x) = x$, $[0, 2]$, $n = 2$.

   (b) $f(x) = x^2$, $[0, 2]$, $n = 4$.

   (c) $f(x) = 8\sqrt{x}$, $[0, 3]$, $n = 100$. Use the calculator program.

2. Approximate the integral
$$\int_1^4 x^{-2} \, dx$$
   using the Left-Right Sums calculator program with $n = 150$ subintervals. Note that the function $\frac{1}{x^2}$ is decreasing on [1,4] so that the left sums gives you the overestimate and the right sum the underestimate.

3. Approximate the following definite integrals using your calculator program to the first two nonzero digits.

   (a) $\int_0^2 \ln(x^2 + 1) \, dx$ (b) $\int_1^3 \frac{e^{2x}}{x} \, dx$

4. The speed of a runner increases during the first three seconds of a race. His speed (in meters per second) at half-second intervals is given in the table bellow. Find lower and upper estimates for the distance he traveled during the first three seconds.

time	0	0.5	1	1.5	2	2.5	3
speed	0	3.1	5.4	7.4	8.9	9.2	10.1

5. A chemical reaction produces a compound X with a rate of 23, 19, 12, 11, 9, 5, 2 liters per second at time intervals spaced by 1 second. Approximate the total volume of the compound X produced in the 6 seconds for which the rate is given using the left and the right sums.

Solutions.

1. (a) $n = 2$ and

x	0	1	2
y	0	1	2

thus $L = \frac{2-0}{2}(0+1) = 1$ and $R = \frac{2-0}{2}(1+2) = 3$.

(b) $n = 4$, and

x	0	0.5	1	1.5	2
y	0	0.25	1	2.25	4

thus $L = \frac{2-0}{4}(0+0.25+1+2.25) = \frac{3.5}{2} = 1.75$ and
$R = \frac{2-0}{4}(0.25+1+2.25+4) = \frac{7.5}{2} = 3.75$.

(c) $f(x) = 8\sqrt{x}$, $a = 0$, $b = 3$ and $n = 100$. Obtain that $L = 27.496$ and $R = 27.912$.

2. Use the program with $f(x) = x^{-2}$, $a = 1$, $b = 4$ and $n = 150$. Obtain that $L = .759$ and $R = .741$. Thus $.741 \le \int_1^4 \frac{1}{x^2}\,dx \le .759$.

3. (a) $f(x) = \ln(x^2+1)$, $a = 0$ and $b = 2$. With $n = 100$ for example, both the left sum and the right sum sound to 1.4. Thus, $\int_0^2 \ln(x^2+1)\,dx \approx 1.4$.

(b) $f(x) = \frac{e^{2x}}{x}$, $a = 1$, and $b = 3$. With $n = 300$ for example, both the left sum and the right sum round to 81. Thus, $\int_1^3 \frac{e^{2x}}{x}\,dx \approx 81$.

4. Similarly as in Example 2, the left and the right sum provide the estimates for distance traveled. In this problem, $a = 0$, $b = 3$, $n = 6$ (careful: n is not 7). The left sum is $L = \frac{3-0}{6}(0+3.1+5.4+7.4+8.9+9.2) = \frac{34}{2} = 17$ and the right sum is $L = \frac{3-0}{6}(3.1+5.4+7.4+8.9+9.2+10.1) = \frac{44.1}{2} = 22.05$. Thus, 22.05 meters is the upper and 17 meters is the lower estimate for the distance.

5. With the restrictions of the measurements above, we approximate the rate during each second with a constant. Thus the volume added in each second can be approximated with *the product of the rate of increase and time.* If the rate values are represented as dependent variable values and time values as the independent variable values, the product of the rate and time corresponds exactly to area of rectangles. Since the rate values are strictly decreasing and positive, the left sum provides an overestimate and the right sum the underestimate of the total volume produced.

Adding the time values to the given table, we obtain the following.

time (sec.)	0	1	2	3	4	5	6
rate (l/sec.)	23	19	12	11	9	5	2

Thus, $a = 0$, $b = 6$ and $n = 6$. $L = \frac{6-0}{6}(23 + 19 + 12 + 11 + 9 + 5) = 79$ liters, $R = \frac{6-0}{6}(19 + 12 + 11 + 9 + 5 + 2) = 58$ liters.