Calculus 2
Lia Vas

Improper Integrals

The integral $\int_a^b f(x)\,dx$ is improper if it is of one of the following **three types**:

1. At least one of the bounds is positive or negative infinity.
2. The function $f(x)$ is not defined or is discontinuous at at least one of the bounds.
3. The function $f(x)$ is not defined or is discontinuous at $x = c$ and $a \le c \le b$. Then reduce the integral to the sum of two type 2 integrals by

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Improper Proper

Geometrically, an improper integral represents the area under a curve that is not necessarily bounded. An unbounded area can still be **finite** or it can be **infinite**. These two scenarios correspond to the improper integral being convergent or divergent:

- An improper integral is **convergent** if it is equal to a real number.
- An improper integral is **divergent** if it is positive or negative infinity or the value of the integral does not exist.

Evaluating an improper integral. Let us assume that $F(x)$ is an antiderivative of $f(x)$ and let us consider three types separately.

1.
$$\int_a^{\infty} f(x)\, dx = \lim_{t\to\infty} F(x) - F(a).$$

Note that "plugging the upper bound in $F(x)$" boils down to evaluating the limit of $F(x)$ when $x \to \infty$. If this limit is a finite number, the integral is convergent. Otherwise, it is divergent.

2. $\int_a^b f(x)\, dx$ if $f(x)$ is not defined at $x = b$ can be evaluated as
$$\int_a^b f(x)\, dx = \lim_{t\to b^-} F(x) - F(a).$$
Note that "plugging the upper bound in $F(x)$" boils down to evaluating the limit of $F(x)$ when $x \to b^-$. If this limit is a finite number, the integral is convergent. Otherwise, it is divergent.
$\int_a^b f(x)\, dx$ if $f(x)$ is not defined at $x = a$ can be evaluated as
$$\int_a^b f(x)\, dx = F(b) - \lim_{t\to a^+} F(x).$$
Note that "plugging the lower bound in $F(x)$" boils down to evaluating the limit of $F(x)$ when $x \to a^+$. If this limit is a finite number, the integral is convergent. Otherwise, it is divergent.

3. Reduce a type 3 integral to a sum of type 1 or type 2 integrals.

If an integral is improper for more than one reason, decompose it into a sum of integrals each of which is of type 1 or type 2. For example, the integral $\int_{-3}^{\infty} \frac{1}{x} dx$ is improper because $x = 0$ is between the bounds and because the upper bound is ∞. It decomposes to a sum of types 1 and 2 as follows, where the bound 1 in the second and the third summand can be replaced by any other positive number.
$$\int_{-3}^{\infty} \frac{1}{x} dx = \int_{-3}^{0} \frac{1}{x} dx + \int_{0}^{1} \frac{1}{x} dx + \int_{1}^{\infty} \frac{1}{x} dx$$
Similarly, the integral $\int_{-\infty}^{\infty} e^x dx$, for example, can be decomposed into a sum of type 1 integrals as follows.
$$\int_{-\infty}^{\infty} e^x dx = \int_{-\infty}^{0} e^x dx + \int_{0}^{\infty} e^x dx$$
Any other number besides 0 can be used as the upper bound in the first and the lower bound in the second summand.

Practice Problems.

1. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

   (a)
$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

   (b)
$$\int_1^{\infty} \frac{1}{x}\, dx$$

   (c)
$$\int_{1000}^{\infty} \frac{1}{x}\, dx$$

(d)

$$\int_0^1 \frac{1}{x^2}\, dx$$

(e)

$$\int_0^\infty \frac{1}{x^2}\, dx$$

(f)

$$\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}}\, dx$$

2. Find an **error** in the following reasoning

$$\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{-1}^1 = -1 - 1 = -2.$$

3. Sketch the region and find its area (if the area is finite).

   (a) $x \geq 3$, $0 \leq y \leq \frac{1}{(x-2)^2}$

   (b) $x \geq 0$, $0 \leq y \leq \frac{1}{(x-2)^2}$

Solutions.

1. (a) This integral is improper because the upper bound is ∞. Evaluate it as follows. $\int_1^\infty \frac{1}{x^2}\, dx = \int_1^\infty x^{-2} dx = \frac{x^{-1}}{-1}|_1^\infty = \frac{-1}{x}\Big|_1^\infty = \lim_{x\to\infty} \frac{-1}{x} - (\frac{-1}{1}) = \frac{1}{\infty} + 1 = 0 + 1 = 1$. We have a finite number as an answer. This means that the integral is convergent.

   We can also interpret this answer to mean that the area between $\frac{1}{x^2}$ and x-axis from 1 to ∞ is finite.

   (b) $\int_1^\infty \frac{1}{x}\, dx = \ln x|_1^\infty = \lim x \to \infty \ln x - \ln 1 = \infty - 0 = \infty$. Thus, the integral is divergent.

   We can also interpret this answer to mean that the area between $\frac{1}{x}$ and x-axis from 1 to ∞ is not finite.

   (c) $\int_{1000}^\infty \frac{1}{x}\, dx = \ln x|_{1000}^\infty = \lim x \to \infty \ln x - \ln 1000 = \infty - 6.91 = \infty$. Thus, the integral is divergent.

   Although changing the lower bound from 1 in the previous problem to 1000 in this problem seem like it significantly reduced the size of the area under the curve, the area still remains infinite.

   (d) This integral is improper because the function $\frac{1}{x^2}$ is not defined at the lower bound $x = 0$. So, this is a type 2 integral. Evaluate it as follows. $\int_0^1 \frac{1}{x^2}\, dx = \frac{x^{-1}}{-1}\Big|_0^1 = \frac{-1}{1} - \lim_{x\to 0^+} \frac{-1}{x} = -1 - (-\infty) = \infty$. So, the integral is divergent.

   We can also interpret this answer to mean that the area between $\frac{1}{x^2}$ and x-axis from 0 to 1 is not finite.

   (e) The integral is improper both because ∞ is the upper bound and because the function $\frac{1}{x^2}$ is not defined at the lower bound $x = 0$. So, we need to decompose it into a sum of a type 2 and a type 1 integral $\int_0^a + \int_a^\infty$ for any positive a. For example, taking $a = 1$ produces $\int_0^1 \frac{1}{x^2} dx + \int_1^\infty \frac{1}{x^2} dx$. The first integral is divergent by the previous problem. Thus, regardless of the fact that the second is convergent (it is equal to 1 by the first problem), the sum of the two integrals is not finite and the initial integral is divergent.

We can also interpret this answer to mean that the area between $\frac{1}{x^2}$ and x-axis from 0 to ∞ is not finite.

(f) The integral is improper because $\frac{1}{\sqrt[3]{x^2}}$ is not defined at $x = 0$ and 0 is between the bounds -1 and 1. Write the integral as a sum of two integrals of type 2. For example $\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\, dx = \int_{-1}^{0} \frac{1}{\sqrt[3]{x^2}}\, dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x^2}}\, dx = 3x^{1/3}\Big|_{-1}^{0} + 3x^{1/3}\Big|_{0}^{1} = 3(0) - 3(-1) + 3(1) - 3(0) = 3 + 3 = 6$. So, the integral is convergent.

2. Note that the integral is improper because 0 is between the bounds -1 and 1. So, it has to be separated as the sum of two type 2 improper integrals

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx.$$

Both integrals are divergent (see problem 1 (c) for the second one, the first one can be evaluated similarly). So, the initial integral is divergent as well.

3. (a) The condition $x \geq 3$ indicated the bounds of the integration $3 \leq x < \infty$. The second condition indicate the lower and upper y-curves: $y = 0$ is the lower and $y = \frac{1}{(x-2)^2}$ is the upper curve. Thus, the area A can be found as

$$A = \int_{3}^{\infty} \left(\frac{1}{(x-2)^2} - 0 \right) dx = \int_{3}^{\infty} \frac{1}{(x-2)^2} dx.$$

Write the function as $(x-2)^{-2}$ so that you can use the power rule. Obtain that

$$A = \frac{(x-2)^{-1}}{-1}\Big|_{3}^{\infty} = \frac{-1}{x-2}\Big|_{3}^{\infty} = \frac{-1}{\infty} - \frac{-1}{1} = 0 + 1 = 1.$$

So, the area is finite and it is equal to 1.

(b) Similarly to the previous problem, the area A can be found as $A = \int_0^{\infty} \frac{1}{(x-2)^2} dx$. Note that this integral is improper both because of the infinity in the bounds and because the value $x = 2$ at which the function $\frac{1}{(x-2)^2}$ is not defined, is between the bounds. So, you need to write this integral as a sum of improper integrals, each of which will be either type 1 or 2. For example, you can do that as follows.

$$A = \int_{0}^{2} \frac{1}{(x-2)^2} dx + \int_{2}^{3} \frac{1}{(x-2)^2} dx + \int_{3}^{\infty} \frac{1}{(x-2)^2} dx.$$

The first integral evaluates as follows.

$$\int_{0}^{2} \frac{1}{(x-2)^2} dx = \frac{-1}{x-2}\Big|_{0}^{2} = \lim_{x \to 2^-} \frac{-1}{x-2} - \frac{-1}{-2} = \frac{-1}{-\infty} - \frac{1}{2} = \infty.$$

Even without evaluating the remaining integrals, you can conclude that the area is not finite based just on this first integral. If you evaluate the second integral, you should get ∞ as well. The third is equal to 1 by the previous problem.