Calculus 2
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Improper Integrals

The integral $\int_a^b f(x)\,dx$ is improper if it is of one of the following **three types**:

1. At least one of the bounds is positive or negative infinity.
2. The function $f(x)$ is not defined or is discontinuous at at least one of the bounds.
3. The function $f(x)$ is not defined or is discontinuous at $x = c$ and $a \leq c \leq b$. Then reduce the integral to the sum of two type 2 integrals by

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Improper   Proper

Geometrically, an improper integral represents the area under a curve that is not necessarily bounded. An unbounded area can still be **finite** or it can be infinite. These two scenarios correspond to the improper integral being convergent or divergent:

- An improper integral is **convergent** if it is equal to a real number.
- An improper integral is **divergent** if it is positive or negative infinity or the value of the integral does not exist.

Evaluating an improper integral. Let us assume that $F(x)$ is an antiderivative of $f(x)$ and let us consider three types separately.

1.
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} F(x) - F(a).$$

Note that "plugging the upper bound in $F(x)$" boils down to evaluating the limit of $F(x)$ when $x \to \infty$. If this limit is a finite number, the integral is convergent. Otherwise, it is divergent.

2. $\int_a^b f(x)\,dx$ if $f(x)$ is not defined at $x = b$ can be evaluated as

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} F(x) - F(a).$$

Note that "plugging the upper bound in $F(x)$" boils down to evaluating the limit of $F(x)$ when $x \to b^-$. If this limit is a finite number, the integral is convergent. Otherwise, it is divergent.

$\int_a^b f(x)\,dx$ if $f(x)$ is not defined at $x = a$ can be evaluated as

$$\int_a^b f(x)\,dx = F(b) - \lim_{t\to a^+} F(x).$$

Note that "plugging the lower bound in $F(x)$" boils down to evaluating the limit of $F(x)$ when $x \to a^+$. If this limit is a finite number, the integral is convergent. Otherwise, it is divergent.

3. Reduce a type-3 integral to a sum of type 1 or 2 integrals.

Practice Problems.

a) Determine whether each integral is convergent or divergent. Evaluate those that are convergent. **Note:** problems 6. 7. and 8. require <u>integration by parts</u>. Do these problems only after this method has been covered.

1.
$$\int_1^\infty \frac{1}{x^2}\,dx$$

2.
$$\int_1^\infty \frac{1}{x}\,dx$$

3.
$$\int_0^1 \frac{1}{x^2}\,dx$$

4.
$$\int_0^\infty \frac{1}{x^2}\,dx$$

5.
$$\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}}\,dx$$

6.
$$\int_0^\infty xe^{-2x}\,dx$$

7.
$$\int_{-\infty}^\infty xe^{-2x}\,dx$$

8.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

b) Find the error in the following reasoning $\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}|_{-1}^{1} = -1-1 = -2$.

c) Sketch the region and find its area (if the area is finite). **Note:** problem 1. requires integration by parts. Do this problem only after this method has been covered.

1. $x \leq 0$, $xe^x \leq y \leq 0$.
2. $x \geq 3$, $0 \leq y \leq \frac{1}{(x-2)^2}$
3. $x \geq 0$, $0 \leq y \leq \frac{1}{(x-2)^2}$

Solutions. a)

1. $\int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}dx = \frac{x^{-1}}{-1}|_1^\infty = \frac{-1}{x}|_1^\infty = \lim_{x\to\infty}\frac{-1}{x} - (\frac{-1}{1}) = \frac{1}{\infty}+1 = 0+1 = 1$. We have a finite number as an answer. This means that the area under the curve $\frac{1}{x^2}$ from 1 to ∞ is finite and that the integral is convergent.

2. $\int_1^\infty \frac{1}{x}\,dx = \ln x|_1^\infty = \lim x \to \infty \ln x - \ln 1 = \infty - 0 = \infty$. Thus, the integral is divergent.

3. Note that this is a type 2 integral: it is improper because $\frac{1}{x^2}$ is not defined at the lower bound $x=0$. $\int_0^1 \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}|_0^1 = \frac{-1}{1} - \lim_{x\to 0^+}\frac{-1}{x} = -1-(-\infty) = \infty$. So, the integral is divergent.

4. Note that this is a sum of a type 1 and a type 2 integrals: it is improper both because of the ∞ and because of 0 in the bounds. Separate it as a sum of integrals $\int_0^a + \int_a^\infty$ for any positive a. For example, you can take $a=1$ you have that $\int_0^1 \frac{1}{x^2}dx + \int_1^\infty \frac{1}{x^2}dx$. The first integral is divergent by the previous problem. Thus, regardless of the fact that the second is convergent (it is equal to 1 by the first problem), the sum of the two integrals is not finite.

   You can think of the sum of the two integrals in this case as the sum of two areas under $\frac{1}{x^2}$. The first, from 0 to 1, is not finite and the second, from 1 to ∞ is equal to 1. The sum $\infty + 1 = \infty$. So, the integral $\int_0^\infty \frac{1}{x^2}\,dx$ is not finite – it is divergent.

5. This is a type 3 integral: it is improper because $\frac{1}{\sqrt[3]{x^2}} = x^{-2/3}$ is not defined at 0 and 0 is between the bounds -1 and 1. Separate the integral into the sum of two integrals of type 2 $\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}}\,dx = \int_{-1}^0 \frac{1}{\sqrt[3]{x^2}}\,dx + \int_0^1 \frac{1}{\sqrt[3]{x^2}}\,dx = 3x^{1/3}|_{-1}^0 + 3x^{1/3}|_0^1 = 3(0)-3(-1)+3(1)-3(0) = 3+3 = 6$. So, the integral is convergent.

6. Find the antiderivative first. Note that you can do that using the by-parts method with $u=x$ and $dv = e^{-2x}dx$. Obtain that the antiderivative $\frac{-1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} = \frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}}$. Thus $\int_0^\infty xe^{-2x}\,dx = \frac{-x}{2e^{2x}}|_0^\infty - \frac{1}{4e^{2x}}|_0^\infty$. For the value at the upper bound in the first part, consider $\lim_{x\to\infty}\frac{-x}{2e^{2x}}$. Use L'Hopital's rule to get $\lim_{x\to\infty}\frac{-1}{4e^{2x}} = \frac{-1}{\infty} = 0$. The value at the upper bound in the second part is $\lim_{x\to\infty}\frac{1}{4e^{2x}} = \frac{1}{\infty} = 0$.

   Thus, the integral is $\frac{-x}{2e^{2x}}|_0^\infty - \frac{1}{4e^{2x}}|_0^\infty = (0-0)-(0-\frac{1}{4}) = \frac{1}{4}$. The integral is equal to a finite number and so it is convergent.

7. Note that this is a sum of two type integrals: $\int_{-\infty}^{\infty} xe^{-2x}\ dx = \int_{-\infty}^{0} xe^{-2x}\ dx + \int_{0}^{\infty} xe^{-2x}\ dx$. Note that you can use any other number instead of 0 for the upper bound in the first and the lower bound in the second part. However, given the previous problem, we can use 0. By the previous problem, the second integral is convergent. The first one is $\left(\frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}}\right)|_{-\infty}^{0} = \frac{-(2x+1)}{4e^{2x}}|_{-\infty}^{0} = \frac{-1}{4} - \frac{\infty}{0^+}$. Here you can think of $\frac{\infty}{0^+}$ as $\infty \cdot \frac{1}{0^+} = \infty \cdot \infty = \infty$. Thus, the first integral is $\frac{-1}{4} - \infty = -\infty$ and so it is not finite. So, the sum of two integrals is divergent.

8. Find the antiderivative first. Note that you can do that using the by-parts method with $u = \ln x$ and $dv = \frac{1}{x^2}dx$. Obtain that the antiderivative $\frac{-\ln x}{x} - \frac{1}{x}$. Thus $\int_1^\infty \frac{\ln x}{x^2}\ dx = \frac{-\ln x}{x}|_1^\infty - \frac{1}{x}|_1^\infty$. For the value at the upper bound in the first part, consider $\lim_{x\to\infty} \frac{-\ln x}{x}$. Use L'Hopital's rule to get $\lim_{x\to\infty} \frac{-1/x}{1} = \frac{-1}{\infty} = 0$. The value at the upper bound in the second part is $\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$. Thus, the integral is $\frac{-\ln x}{x}|_1^\infty - \frac{1}{x}|_1^\infty = (0-0) - (0-1) = 1$. The integral is equal to a finite number and so it is convergent.

b) Note that the integral is improper because 0 is between the bounds -1 and 1. So, it has to be separated as the sum of two type 2 improper integrals $\int_{-1}^{1} \frac{1}{x^2}\ dx = \int_{-1}^{0} \frac{1}{x^2}\ dx + \int_{0}^{1} \frac{1}{x^2}\ dx$. Both integrals are divergent (see problem 3 for the second one, the first one can be evaluated similarly). So, the initial integral is divergent as well.

c)

1. The condition $x \leq 0$ indicated the bounds of the integration $-\infty < x \leq 0$. The second condition indicate the lower and upper y-curves: $y = xe^x$ is the lower and $y = 0$ is the upper curve. Thus, the area A can be found as $\int_{-\infty}^{0}(0 - xe^x)dx = \int_{-\infty}^{0} -xe^x dx$. Use the integration by parts to find the antiderivative $-xe^x + e^x$. So $A = -xe^x|_{-\infty}^0 + e^x|_{-\infty}^0$. For the first part, note that $\lim_{x\to-\infty} -xe^x = -\infty \cdot 0$. This requires the use of L'Hopital's rule. Write your function as $\frac{-x}{e^{-x}}$ to obtain $\frac{\infty}{\infty}$ form. Then use the rule to get $\lim_{x\to-\infty} \frac{-1}{-e^{-x}} = \frac{1}{\infty} = 0$.

   For the second part, note that $\lim_{x\to-\infty} e^x = 0$.

   Thus, $A = -xe^x|_{-\infty}^0 + e^x|_{-\infty}^0 = (0-0) + (1-0) = 1$. So, the area is finite and it is equal to 1.

2. The condition $x \geq 3$ indicated the bounds of the integration $3 \leq x < \infty$. The second condition indicate the lower and upper y-curves: $y = 0$ is the lower and $y = \frac{1}{(x-2)^2}$ is the upper curve. Thus, the area A can be found as $\int_3^\infty (\frac{1}{(x-2)^2} - 0)dx = \int_3^\infty \frac{1}{(x-2)^2}dx = \frac{(x-2)^{-1}}{-1}|_3^\infty = \frac{-1}{x-2}|_3^\infty = \frac{-1}{\infty} - \frac{-1}{1} = 0 + 1 = 1$. So, the area is finite and it is equal to 1.

3. Similarly to the previous problem, the area A can be found as $A = \int_0^\infty \frac{1}{(x-2)^2}dx$. Note that this integral is improper both because of the infinity in the bounds and because the value $x = 2$ at which the function $\frac{1}{(x-2)^2}$ is not defined, is between the bounds. So, you need to write this integral as a sum of improper integrals, each of which will be either type 1 or 2. For example, you can do that as follows.

$$A = \int_0^2 \frac{1}{(x-2)^2}dx + \int_2^3 \frac{1}{(x-2)^2}dx + \int_3^\infty \frac{1}{(x-2)^2}dx.$$

   The first integral $\int_0^2 \frac{1}{(x-2)^2}dx$ is equal to $\frac{-1}{x-2}|_0^2 = \lim_{x\to2^-} \frac{-1}{x-2} - \frac{-1}{-2} = \frac{-1}{-\infty} - \frac{1}{2} = \infty$. Even without evaluating the remaining integrals, you can conclude that the area is not finite based just on this first integral. If you evaluate the second integral, you should get ∞ as well. The third is equal to 1 by the previous problem.