Calculus 2
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The Indefinite Integral – Review

The indefinite integral solves the following general problem.

> Given a function $f(x)$, find function y such that $y' = f(x)$.

Finding a solution of this general problem requires the use of the *inverse of derivation*, that is the *process that produces a function whose derivative is known.* This process is known as the **antidifferentiation** and the outcome of this process is known as an **antiderivative**. Thus,

> A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$.

The antidifferentiation is relevant when the rate of change of a quantity can be measured and the quantity size itself needs to be determined from the rate. if velocity is known and we need to determine the function computing the distance traveled, the antiderivation is needed.

If $F(x)$ is an antiderivative of $f(x)$ then so is $F(x)+c$. Indeed if $\frac{d}{dx}F(x) = f(x)$ then $\frac{d}{dx}(F(x)+c) = f(x)$ as well since the derivative of a constant is zero. Conversely, in Calculus 1 you have seen that any antiderivative of $f(x)$ has to have the form $F(x) + c$ for some constant c.

The collection of *all* antiderivatives of a function $f(x)$ is called the **indefinite integral** and it is denoted by

$$\int f(x)\, dx$$

Thus, we have the following.

> If $F(x)' = f(x)$ then $\int f(x)dx = F(x) + c$.

Let us recall some rules of integration.

> **The Sum Rule.** $\int(f(x) + g(x))dx = \int f(x)\, dx + \int g(x)dx$
> **The Constant Multiple Rule.** $\int kf(x)dx = k \int f(x)\, dx$
> **The Power Rule.** $\int x^n dx = \frac{1}{n+1}x^n + c$

These two rules enables us to find antiderivative of any polynomial function by **integrating term by term**. It is important to keep the following in mind.

1. Keep carrying *both* the integral sign $\int$ and the symbol dx until you use an integration rule (just the power rule for the time being) which eliminates $\int$ and dx *simultaneously.*

2. When $\int$ and dx are eliminated, *do not forget to add a constant* to your answer.

Keep in mind the following algebra rules can be handy when integrating non-polynomial power functions.

$$\frac{1}{x^n} = x^{-n} \qquad\qquad \sqrt[n]{x} = x^{1/n}$$

Applications.

Finding an antiderivative applies to every situation when the rate is given and the function with the given rate is needed. For example, recall that the velocity is the derivative of the distance function thus, the *distance is an antiderivative of the velocity.* Similarly, since the acceleration is the derivative of velocity, the *velocity is an antiderivative of the acceleration.*

Practice Problems.

1. Evaluate the following integrals.

   (a) $\int (x^5 + 2)\, dx$ (b) $\int (4x^3 + 6x^2 - 4x + 3)\, dx$ (c) $\int (\sqrt{x} - \frac{4}{x^2})\, dx$

   (d) $\int \frac{1+x^3}{x^2}\, dx$ (e) $\int \frac{8\sqrt[3]{x}+x^2}{4}\, dx$ (f) $\int \frac{x+3}{\sqrt{x}}\, dx$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

   (a) $f'(x) = \frac{1}{\sqrt[3]{x}}$ and $f(8) = 9$

   (b) $f'(x) = 5\sqrt{x^3} + 3$ and $f(1) = 4$

3. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{2}$ where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 2$ seconds, the position function $s(t) = 5$ feet, determine the position function $s(t)$.

4. A stone is being thrown up in the air with initial velocity of 5 m/s. Determine the time the object hits the ground and find the speed at the time of the impact. Determine also the maximal height it reaches. You can neglect the drag force and assume that the gravitational force is the only force that acts on an object producing the constant acceleration of 9.8 m/s^2.

Solutions.

1. (a) $\int (x^5 + 2)\, dx = \int x^5 dx + \int 2dx = \frac{1}{6}x^6 + 2x + c.$

   (b) $\int (4x^3 + 6x^2 - 4x + 3)\, dx = 4\frac{1}{4}x^4 + 6\frac{1}{3}x^3 - 4\frac{1}{2}x^2 + 3x + c = x^4 + 2x^3 - 2x^2 + 3x + c.$

   (c) $\int (x^{1/2} - 4x^{-2})\, dx = \frac{2}{3}x^{3/2} + \frac{4}{x} + c$

   (d) $\int \frac{1+x^3}{x^2}\, dx = \int (\frac{1}{x^2} + \frac{x^3}{x^2})\, dx = \int (x^{-2} + x)\, dx = -\frac{1}{x} + \frac{x^2}{2} + c$

   (e) $\int \frac{8\sqrt[3]{x}+x^2}{4}\, dx = \int (2x^{1/3} + \frac{1}{4}x^2)dx = 2\,\frac{3}{4}x^{4/3} + \frac{1}{4} \cdot \frac{1}{3}x^3 + c = \frac{3}{2}\sqrt[3]{x^4} + \frac{1}{12}x^3 + c$

   (f) $\int \frac{x+3}{\sqrt{x}}\, dx = \int (\frac{x}{\sqrt{x}} + \frac{3}{\sqrt{x}})dx = \int x^{1/2}\, dx + \int 3x^{-1/2}\, dx = \frac{1}{3/2}x^{3/2} + 3\frac{1}{1/2}x^{1/2} = \frac{2}{3}\sqrt{x^3} + 6\sqrt{x} + c$

2. (a) $f(x) = \int f'(x)\ dx = \int x^{-1/3} dx = \frac{3}{2}\sqrt[3]{x^2} + c$. Using $f(8) = 9$ to solve for c, you have that $9 = \frac{3}{2}\sqrt[3]{8^2} + c = \frac{3}{2} \cdot 4 + c = 6 + c \Rightarrow 9 = 6 + c \Rightarrow c = 3$. Thus, $f(x) = \frac{3}{2}\sqrt[3]{x^2} + 3$.

   (b) $f(x) = \int f'(x)\ dx = \int (5x^{3/2} + 3)dx = 5\,\frac{2}{5}x^{5/2} + 3x + c = 2\sqrt{x^5} + 3x + c$. Using $f(1) = 4$ to solve for c, you have that $4 = 2\sqrt{1^5} + 3(1) + c = 2 + 3 + c = 5 + c \Rightarrow 4 = 5 + c \Rightarrow c = -1$. Thus, $f(x) = 2\sqrt{x^5} + 3x - 1$.

3. Recall that $s(t) = \int v(t)dt$. Thus $s(t) = \frac{1}{2}\,\frac{t^2}{2} + c = \frac{t^2}{4} + c$. Using that $s(2) = 5$, we have that $5 = \frac{2^2}{4} + c = 1 + c \Rightarrow c = 4$. Thus $s(t) = \frac{t^2}{4} + 4$.

4. The velocity is decreasing and positive during the time the object goes up. When the object starts falling, the velocity is negative and becoming more and more negative, that is, it is also decreasing. Thus the acceleration is negative throughout the motion. $a(t) = v'(t) = -9.8 \Rightarrow v(t) = \int -9.8dt = -9.8t + c$. Since the initial velocity is 5, $v(0) = 5 \Rightarrow 5 = -9.8(0) + c \Rightarrow c = 5$ and so $v(t) = s'(t) = -9.8t + 5$. Obtain the distance function as $s(t) = \int(-9.8t + 5)dt = -4.9t^2 + 5t + c$. Since $s(0) = 0$, we have that $c = 0$ and so $s(t) = -4.9t^2 + 5t$.

   The object hits the ground when $s(t) = 0$ and $t > 0$. Since $s(t)$ factors as $t(-4.9t + 5)$ the nonzero solution is $5 = 4.9t \Rightarrow t = 1.02$ seconds. The velocity at that time is $v(1.02) = -5$ m/s so the final speed is the same as the initial speed.

   The maximal height can be obtained as the value of s at the critical point (i.e. time when the velocity is zero). $v(t) = -9.8t + 5 = 0 \Rightarrow t = \frac{5}{9.8} = 0.51$ second.

Substitution – Review

Many integrals cannot be evaluated using the above rules (and many others cannot be evaluated even by methods we shall cover in remainder of the course). Still there is a class of integrals which can be evaluated using a method known as the **substitution**. Namely, if the integrand is a constant multiple of a function of the form $f(g(x))\ g'(x)$ and an antiderivative of $f(x)$ can be found to be $F(x)$, then $F'(x) = f(x)$ and the Chain Rule applied to the function $F(g(x))$ produces the function $F'(g(x))g'(x) = f(g(x))g'(x)$. This implies that $F(g(x)))$ is an antiderivative of $f(g(x))g'(x)$.

$$\frac{d}{dx}(F(g(x))) = f(g(x))\ g'(x) \quad \Rightarrow \quad \int f\,(g(x))\ g'(x)\ dx = F(g(x)) + c.$$

The last formula is known as the Substitution Rule. You can think of the function $u = g(x)$ as the inner function and consider the Substitution Rule to be applicable if the integrand is of the form

constant multiple of	$f(g(x))$	$\cdot$	$g'(x)$
	the composite	$\cdot$	the derivative of the inner

To check if the integrand is of this form and perform substitution, use the following steps.

1. Start by analyzing the integrand, determining whether you can evaluate the integral directly, without the substitution, or, if the substitution is appropriate, follow the steps below.

2. **Identify the inner function** $u = g(x)$ (usually the term under a radical, term in parenthesis, denominator, exponent...).

3. **Find the differential** $du = g'(x)dx$. and solve for $dx = \frac{du}{g'(x)}$.

4. **Substitute** the inner function $g(x)$ with u and dx with dx from previous step.

5. The substitution is successful if there are **<u>no terms with x left</u>** in the integrand. You may need to simplify the integrand and use the relation $u = g(x)$ again.

6. If you obtain a simpler integrand than the initial one and can evaluate it using the rules of integration, the substitution method worked.

7. Then **integrate** the integrand.

8. Finally, **substitute back** using that $u = g(x)$ so that your final answer is in terms of x again.

In some cases, a bit more delicate situation arises when after picking the inner function for u and making the substitution, the integrand needs to be simplified further in order to complete the transition from x to u. We shall encounter these integrals again when finding the surface area of revolving parametric curve.

Example. Evaluate the following integrals.

(a) $\displaystyle\int x\sqrt{x-3}\,dx$ (b) $\displaystyle\int x^3\sqrt{1+x^2}\,dx$

Solution. (a) Start with $u = x - 3$ and differentiating to get $du = dx$. Substitute and obtain $\int x\sqrt{u}\,du$. Use the formula $u = x - 3$ to express x in terms of u as $x = u + 3$. The integral becomes $\int(u+3)\sqrt{u}du = \int(u^{3/2} + 3u^{1/2})du = \frac{2}{5}u^{5/2} + 3\frac{2}{3}u^{3/2} + c = \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + c$.

(b) Start with picking the inner function $1 + x^2$ for u. $u = x^2 + 1 \Rightarrow du = 2xdx \Rightarrow dx = \frac{du}{2x}$. Substitute and obtain $\int x^3\sqrt{u}\,\frac{du}{2x} = \frac{1}{2}\int x^2\sqrt{u}\,du$. Use the formula $u = x^2 + 1$ to express x^2 in terms of u as $x^2 = u - 1$ and substitute that in the integral as well. The integral becomes $\frac{1}{2}\int(u-1)\sqrt{u}du = \frac{1}{2}\int(u^{3/2} - u^{1/2})du = \frac{1}{2}(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}) + c = \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + c$.

Practice Problems.

1. Determine if the substitution method is appropriate for evaluating the following integrals and, if it is, evaluate them using substitution.

   (a) $\displaystyle\int(3x+5)^6\;\;dx$ (b) $\displaystyle\int(2x+1)^3\,dx$ (c) $\displaystyle\int\frac{x}{(x^2+3)^2}\,dx$

   (d) $\displaystyle\int\frac{(x^2+1)^2}{x^2}\;\;dx$ (e) $\displaystyle\int\frac{x}{(2x+1)^2}\,dx$ (f) $\displaystyle\int\frac{x^2}{\sqrt{x^3-5}}\,dx$

   (g) $\displaystyle\int\frac{1}{\sqrt{x^4+7}}\;\;dx$ (h) $\displaystyle\int\frac{6}{\sqrt[3]{3x+5}}\,dx$ (i) $\displaystyle\int 3x\left((x^2+1)^2 - \sqrt{x^2+1}\right)dx$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

(a) $f'(x) = \sqrt{2x+9}$ and $f(0) = 5$

(b) $f'(x) = \frac{10}{\sqrt{4x+1}}$ and $f(0) = 3$

(c) $f'(x) = x^2\sqrt{4x^4+9x^2}$ and $f(0) = -2$.

3. Suppose that the velocity of an object is given by the function

$$v(t) = \frac{t}{\sqrt{t^2+9}}$$

where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 4$ seconds, the position function $s(t) = 8$ feet, determine the position function $s(t)$.

Solutions.

1. (a) Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int (3x+5)^6\ dx = \int u^6 \frac{du}{3} = \frac{1}{3}\int u^6 du = \frac{1}{3}\frac{u^7}{7} + c = \frac{u^7}{21} + c = \frac{(3x+5)^7}{21} + c$.

   (b) Use substitution $u = 2x+1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int (2x+1)^3\ dx = \int u^3 \frac{du}{2} = \frac{1}{2}\int u^3 du = \frac{1}{2}\frac{u^4}{4} + c = \frac{u^4}{8} + c = \frac{(2x+1)^4}{8} + c$.

   (c) Use substitution $u = x^2+3$. Then $du = 2xdx$ so $dx = \frac{du}{2x}$. $\int \frac{x}{(x^2+3)^2}\ dx = \int \frac{x}{u^2}\frac{du}{2x} = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\frac{1}{-1}u^{-1} + c = \frac{-1}{2u} + c = \frac{-1}{2(x^2+3)} + c$.

   (d) Note that substitution $u = x^2+1$ does not work since there is a remaining term x in denominator after substituting. The integral can be evaluated without substitution since the integrand simplifies as $\frac{(x^2+1)^2}{x^2} = \frac{x^4+2x^2+1}{x^2} = x^2+2+x^{-2}$ so that the integral becomes $\int (x^2+2+x^{-2})dx = \frac{1}{3}x^3 + 2x + \frac{1}{-1}x^{-1} + c = \frac{x^3}{3} + 2x - \frac{1}{x} + c$.

   v Use substitution $u = 2x+1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. Substitute and obtain $\int x(2x+1)^{-2}dx = \int xu^{-2}\frac{du}{2}$. Use the formula $u = 2x+1$ to express x in terms of u as $x = \frac{1}{2}(u-1)$. The integral becomes $\int \frac{1}{2}(u-1)u^{-2}\frac{du}{2} = \frac{1}{4}\int (u^{-1} - u^{-2})du = \frac{1}{4}(\ln|u| - \frac{1}{-1}u^{-1}) + c = \frac{1}{4}\ln|u| + \frac{1}{4u} + c = \frac{1}{4}\ln|2x+1| + \frac{1}{4(2x+1)} + c$.

   (e) Use substitution $u = x^3-5$. Then $du = 3x^2dx$ so $dx = \frac{du}{3x^2}$. $\int \frac{x^2}{\sqrt{x^3-5}}\ dx = \int \frac{x^2}{\sqrt{u}}\frac{du}{3x^2} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\frac{u^{1/2}}{1/2} + c = \frac{2\sqrt{u}}{3} + c = \frac{2\sqrt{x^3-5}}{3} + c$ or $\frac{2}{3}\sqrt{x^3-5} + c$.

   (f) The integral cannot be evaluated by substitution $u = x^4+7$. Substitutions $u = \sqrt{x^4+7}$ or $u = (x^4+7)^{-1/2}$ are equally ineffective.

   (g) Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int \frac{6}{\sqrt[3]{3x+5}}\ dx = 6\int \frac{1}{u^{1/3}}\frac{du}{3} = \frac{6}{3}\int u^{-1/3}du = 2\frac{u^{2/3}}{2/3} + c = 3u^{2/3} + c = 3(3x+5)^{2/3} + c$ or $3\sqrt[3]{(3x+5)^2} + c$.

   (h) Use substitution $u = x^2+1$ so that $du = 2xdx$ and $dx = \frac{du}{2x}$ to obtain $\int 3x(u^2 - u^{1/2})\ \frac{du}{2x} = \frac{3}{2}\int (u^2 - u^{1/2})\ du = \frac{3}{2}(\frac{1}{3}u^3 - \frac{2}{3}u^{3/2}) + c = \frac{1}{2}u^3 - u^{3/2} + c = \frac{1}{2}(x^2+1)^3 - (x^2+1)^{3/2} + c$.

2. (a) $f(x) = \int f'(x)\ dx = \int \sqrt{2x+9}\, dx$. Use substitution $u = 2x+9$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int \sqrt{2x+9}\ dx = \int u^{1/2}\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2} + c = \frac{1}{3}u^{3/2} + c = \frac{1}{3}(2x+9)^{3/2} + c$. Using $f(0) = 5$ to solve for c, you have that $5 = \frac{1}{3}\sqrt{9^3} + c = \frac{27}{3} + c \Rightarrow 5 = 9 + c \Rightarrow c = -4$. Thus, $f(x) = \frac{1}{3}(2x+9)^{3/2} - 4$.

(b) $f(x) = \int f'(x)\,dx = \int \frac{10}{\sqrt{4x+1}}\,dx$. Use substitution $u = 4x+1$. Then $du = 4dx$ so $dx = \frac{du}{4}$. $\int \frac{10}{\sqrt{4x+1}}\,dx = 10\int u^{-1/2}\frac{du}{4} = \frac{10}{4}\frac{u^{1/2}}{1/2} + c = 5u^{1/2} + c = 5\sqrt{4x+1} + c$. Using $f(0) = 3$ to solve for c, you have that $3 = 5\sqrt{0+1} + c = 5 + c \Rightarrow 3 = 5 + c \Rightarrow c = -2$. Thus, $f(x) = 5\sqrt{4x+1} - 2$.

(c) $f(x) = \int x^2\sqrt{4x^4 + 9x^2}dx$. First simplify the integrand as $\int x^2\sqrt{x^2(4x^2+9)}dx =$ $\int x^2\, x\,\sqrt{4x^2+9}dx = \int x^3\sqrt{4x^2+9}dx$. Use the substitution $u = 4x^2+9 \Rightarrow du = 8xdx \Rightarrow dx = \frac{dx}{8x}$. Substitute and obtain $\int x^3\sqrt{u}\,\frac{du}{8x} = \frac{1}{8}\int x^2\sqrt{u}\,du$. Use the formula $u = 4x^2+9$ to express x^2 in terms of u as $x^2 = \frac{1}{4}(u-9)$ and substitute that in the integral as well. The integral becomes $\frac{1}{32}\int(u-9)\sqrt{u}du = \frac{1}{32}\int(u^{3/2} - 9u^{1/2})du = \frac{1}{32}(\frac{2}{5}u^{5/2} - 9\frac{2}{3}u^{3/2}) + c =$ $\frac{1}{80}(4x^2+9)^{5/2} - \frac{3}{16}(4x^2+9)^{3/2} + c$. Since $f(0) = -2$, $-2 = \frac{1}{80}9^{5/2} - \frac{3}{16}9^{3/2} + c =$ $\frac{243}{80} - \frac{81}{16} + c \Rightarrow -2 = \frac{-81}{40} + c \Rightarrow c = \frac{1}{40}$ resulting in $f(x) = \frac{1}{80}(4x^2+9)^{5/2} - \frac{3}{16}(4x^2+9)^{3/2} + \frac{1}{40}$.

3. $s(t) = \int v(t)\;dt = \int \frac{t}{\sqrt{t^2+9}}\,dt$. Use substitution $u = t^2+9$. Then $du = 2tdt$ so $dt = \frac{du}{2t}$. $\int \frac{t}{\sqrt{t^2+9}}\;dt = \int \frac{t}{u^{1/2}}\frac{du}{2t} = \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\frac{u^{1/2}}{1/2} + c = u^{1/2} + c = \sqrt{t^2+9} + c$. Using $s(4) = 8$ to solve for c, you have that $8 = \sqrt{4+9} + c = 5 + c \Rightarrow 8 = 5 + c \Rightarrow c = 3$. Thus, $s(t) = \sqrt{t^2+9} + 3$.