

L'Hôpital's Rule

L'Hôpital's rule is used to convert limits in an indeterminate form to a determinate form. One can apply it in several situations.

Basic Case $\frac{0}{0}$ or $\frac{\infty}{\infty}$. To evaluate a limit $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, consider the limit $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$. If this limit can be determined, then the original limit is equal to it. This fact is usually stated as follows.



L'Hôpital's Rule:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Also note that in some cases the limit $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ may be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again. In this case, try to use the L'Hôpital's rule **again**.

Careful: Don't say that $\frac{0}{0} = 0$.

Don't cancel ∞ in $\frac{\infty}{\infty}$ to get 1.



“Infinity times zero” case. If the limit $\lim_{x \rightarrow a} f(x) \cdot g(x)$ is of the type $0 \cdot \infty$, it can be reduced to the basic case $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Write fg as $\frac{f}{1/g}$ or $\frac{g}{1/f}$ to get the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

“Infinity - infinity” case. Limits $\lim_{x \rightarrow a} f(x) - g(x)$ of the type $\infty - \infty$ can be converted to the basic case by transforming the difference on some way. Useful rule in some cases is that $\log_a x - \log_a y = \log_a \frac{x}{y}$. Other transformations may include using a common denominator, factoring a common term or rationalizing certain expression.

Indeterminate f^g forms. If the limit $\lim_{x \rightarrow a} f(x)^{g(x)}$ is of the type 0^0 , ∞^0 or 1^∞ , you can reduce it to the “infinity times zero” case by doing the following steps.

1. Consider \ln of the limit $\lim_{x \rightarrow a} f(x)^{g(x)}$.

2. Rewrite $\ln f(x)^{g(x)}$ as $g(x) \cdot \ln f(x)$. Note that you can exchange the order of \ln and the limit since $\ln x$ is a continuous function.
3. Obtain the limit $\lim_{x \rightarrow a} g(x) \ln f(x)$ that will be of the “infinity times zero” type and evaluate it. Say you obtain the answer L .
4. Your original limit is equal to e^L . Recall that we changed the original limit by taking \ln of it and now we need to compensate to that change by applying the inverse function e^x of $\ln x$.

Practice Problems. Find the limits.

1.

$$\lim_{x \rightarrow 1} \frac{x^{21} - 1}{x^8 - 1}$$

2.

$$\lim_{x \rightarrow 0} \frac{e^{4x} - 1}{\sin 2x}$$

3.

$$\lim_{x \rightarrow \infty} \frac{e^x}{x}$$

4.

$$\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2}$$

5.

$$\lim_{x \rightarrow \infty} 3xe^{-2x}$$

6.

$$\lim_{x \rightarrow 0^+} x \ln x$$

7.

$$\lim_{x \rightarrow \infty} \ln(x + 2) - \ln(x - 1)$$

8.

$$\lim_{x \rightarrow \infty} \ln(3x^2 + 5) - \ln(2x^2 + 7)$$

9.

$$\lim_{x \rightarrow \infty} x^{1/x}$$

10.

$$\lim_{x \rightarrow 0} (1 - 2x)^{1/x}$$

11.

$$\lim_{x \rightarrow \infty} \left(1 - \frac{5}{x}\right)^{2x}$$

12.

$$\lim_{x \rightarrow 0} (1 + 3x)^{1/x}$$

Solutions.

1. Substituting 1 for x tells you that the limit is of the form $\frac{0}{0}$. Using L'Hôpital: $\lim_{x \rightarrow 1} \frac{x^{21}-1}{x^8-1} = \lim_{x \rightarrow 1} \frac{21x^{20}}{8x^7} = \frac{21}{8}$.
2. Form $\frac{0}{0}$. Using L'Hôpital: $\lim_{x \rightarrow 0} \frac{e^{4x}-1}{\sin 2x} = \lim_{x \rightarrow 0} \frac{4e^{4x}}{2 \cos 2x} = \frac{4(1)}{2(1)} = 2$.
3. Form $\frac{\infty}{\infty}$. Using L'Hôpital: $\lim_{x \rightarrow \infty} \frac{e^x}{x} = \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty$.
4. Form $\frac{0}{0}$. Using L'Hôpital: $\lim_{x \rightarrow 0} \frac{1-\cos 3x}{x^2} = \lim_{x \rightarrow 0} \frac{3 \sin 3x}{2x}$. Note that this is also of the form $\frac{0}{0}$. Use the L'Hôpital's rule **again**. Obtain $\lim_{x \rightarrow 0} \frac{3 \sin 3x}{2x} = \lim_{x \rightarrow 0} \frac{9 \cos 3x}{2} = \frac{9}{2}$.
5. Form $\infty \cdot 0$. Note that the term e^{-2x} can be written as $\frac{1}{e^{2x}}$ and then the limit becomes of the form $\frac{\infty}{\infty}$. Thus, $\lim_{x \rightarrow \infty} 3xe^{-2x} = \lim_{x \rightarrow \infty} \frac{3x}{e^{2x}}$ using L'Hôpital, this becomes $\lim_{x \rightarrow \infty} \frac{3}{2e^{2x}} = \frac{1}{\infty} = 0$.
6. Form $-\infty \cdot 0$. You may be debating if you should write the function $x \ln x$ as $\frac{\ln x}{1/x}$ or as $\frac{x}{1/\ln x}$. Applying the rule in both cases. You will see that it gives you shorter expression in the first case, so this is the way you may want to go. Thus, $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-x^{-2}} = \lim_{x \rightarrow 0^+} -x^{-1+2} = \lim_{x \rightarrow 0^+} -x = 0$.
7. The limit is of the form $\infty - \infty$. Write the function $\ln(x+2) - \ln(x-1)$ as $\ln \frac{x+2}{x-1}$ and note that the quotient in the argument of \ln is of the type $\frac{\infty}{\infty}$. Using the L'Hôpital's rule for this quotient gives you $\lim_{x \rightarrow \infty} \frac{x+2}{x-1} = \frac{1}{1} = 1$. Thus, $\ln \frac{x+2}{x-1}$ approaches $\ln 1 = 0$ when $x \rightarrow \infty$.
8. The limit is of the form $\infty - \infty$. Write the function $\ln(3x^2+5) - \ln(2x^2+7)$ as $\ln \frac{3x^2+5}{2x^2+7}$ and note that the quotient in the argument of \ln is of the type $\frac{\infty}{\infty}$. Using the L'Hôpital's rule for this quotient gives you $\lim_{x \rightarrow \infty} \frac{3x^2+5}{2x^2+7} = \lim_{x \rightarrow \infty} \frac{6x}{4x} = \frac{6}{4} = \frac{3}{2}$. Thus, $\ln \frac{3x^2+5}{2x^2+7}$ approaches $\ln \frac{3}{2} = 0.405$ when $x \rightarrow \infty$.
9. Take \ln of the given limit: $\ln \lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} \ln x^{1/x} = \lim_{x \rightarrow \infty} \frac{1}{x} \ln x = \lim_{x \rightarrow \infty} \frac{\ln x}{x}$ which is of the form $\frac{\infty}{\infty}$. Using L'Hôpital's rule, get $\lim_{x \rightarrow \infty} \frac{1/x}{1} = \frac{1}{\infty} = 0$. So, the original limit is equal to $e^0 = 1$.
10. Take \ln of the given limit: $\lim_{x \rightarrow 0} \ln(1-2x)^{1/x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1-2x) = \lim_{x \rightarrow 0} \frac{\ln(1-2x)}{x}$ which is of the form $\frac{0}{0}$. Using L'Hôpital's rule, get $\lim_{x \rightarrow 0} \frac{\frac{1}{1-2x}(-2)}{1}$ Plug 0 for x . Get $\frac{\frac{1}{1}(-2)}{1} = -2$. So, the given limit is equal to $e^{-2} = .135$.
11. Take \ln of the given limit: $\lim_{x \rightarrow \infty} \ln(1 - \frac{5}{x})^{2x} = \lim_{x \rightarrow \infty} 2x \ln(1 - \frac{5}{x})$. This is of the form $\infty \cdot 0$. You can keep $2 \ln(1 - \frac{5}{x})$ in the numerator and write x as $1/x$ in the denominator. Thus, we have $\lim_{x \rightarrow \infty} \frac{2 \ln(1 - \frac{5}{x})}{1/x}$ and you can apply the rule now. Obtain $\lim_{x \rightarrow \infty} \frac{2 \cdot \frac{1}{1 - \frac{5}{x}} (-(-5))x^{-2}}{-x^{-2}}$. Note that terms x^{-2} in numerator and denominator cancel and that the term $\frac{5}{x} \rightarrow 0$ for $x \rightarrow \infty$. Thus we have $\frac{2 \cdot \frac{1}{1-0}(5)}{-1} = -10$. So, the given limit is equal to $e^{-10} = 4.5 \cdot 10^{-5}$.
12. Take \ln of the given limit: $\lim_{x \rightarrow 0} \ln(1+3x)^{1/x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+3x) = \lim_{x \rightarrow 0} \frac{\ln(1+3x)}{x}$ which is of the form $\frac{0}{0}$. Using L'Hôpital's rule, get $\lim_{x \rightarrow 0} \frac{\frac{1}{1+3x}(3)}{1} = \frac{1(3)}{1} = 3$. So, the given limit is equal to $e^3 = 20.09$.