Calculus 2
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Linear Differential Equation

A first order differential equation is **linear** if it can be written in the form $a_1(x)y' + a_0(x)y = b(x)$.

Note that if $a_1(x) = 0$, the equation is not differential. So, let us assume that the function $a_1(x)$ is not zero. In this case we can **divide the equation with** $a_1(x)$ and obtain the form

$$y' + P(x)y = Q(x).$$

Note that here $P = a_0/a_1$ and $Q = b/a_1$.

To solve this differential equation you should:

1. Write the equation in the form $y' + P(x)y = Q(x)$.
2. Find the **integrating factor** $I(x) = e^{\int P(x)dx}$ and multiply both sides of the equation with it.
3. Note that the left side is the derivative of the product $I(x) \cdot y$.
4. Integrate both sides. On the left side you will have the product $I(x) \cdot y$.
5. Solve for y.

Practice Problems. Solve the following equations.

1. $y' + 2y = 2e^x, \quad y(0) = 1.$
2. $y' - 2y = x.$
3. $xy' + 2y = x^3.$
4. $x^2y' + xy = 1, \quad y(1) = 2.$
5. $xy' + 2y = \cos x, \quad y(\pi) = 0.$

Solutions.

1. For the equation $y' + 2y = 2e^x$, you have that $P = 2$. Determine the integrating factor as $I = e^{\int 2dx} = e^{2x}$. Multiply the equation by it to get $y'e^{2x} + 2e^{2x}y = 2e^xe^{2x}$. Note that the left side is the derivative of the product ye^{2x} (check: the product rule for ye^{2x} gives you $y'e^{2x} + 2e^{2x}y$ which is exactly the left side). So, the equation becomes $(ye^{2x})' = 2e^{3x}$. Integrate both sides to get $ye^{2x} = \int 2e^{3x}dx \Rightarrow ye^{2x} = \frac{2}{3}e^{3x} + c$. Finally, divide by e^{2x} to get the general solution $y = \frac{\frac{2}{3}e^{3x}+c}{e^{2x}} = \frac{2}{3}e^x + ce^{-2x}$.

   Using the initial condition $y(0) = 1$, you have $1 = \frac{2}{3}e^0 + ce^0 = \frac{2}{3} + c \Rightarrow c = \frac{1}{3}$. Thus the solution is $y = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}$.

2. For the equation $y' - 2y = x$, you have that $P = -2$. *Careful:* don't forget the negative sign. The integrating factor is $I = e^{\int -2dx} = e^{-2x}$. Multiply the equation by it to get $y'e^{-2x} - 2e^{-2x}y = xe^{-2x}$. Note that the left side is the derivative of the product ye^{-2x}. So, the equation becomes $(ye^{-2x})' = xe^{-2x}$. Integrate both sides to get $ye^{-2x} = \int xe^{-2x}dx$. Using the integration by parts with $u = x$ and $dv = e^{-2x}dx$ for the right side, obtain that $ye^{-2x} = \frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c$. Divide by e^{-2x} to get the general solution $y = \frac{\frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c}{e^{-2x}} = \frac{-x}{2} - \frac{1}{4} + ce^{2x}$.

3. *Careful*: before determining P, you have to write the equation in the form $y' + Py = Q$. So, you need to divide by x first. Obtain $y' + \frac{2}{x}y = x^2$. This gives you that $P = \frac{2}{x}$. The integrating factor is $I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$. *Careful*: don't cancel $e^{2\ln x}$ as $2x$.

   Multiply the equation by x^2 to get $y'x^2 + 2xy = x^4$. Note that the left side is the derivative of the product yx^2. So, the equation becomes $(yx^2)' = x^4$. Integrate both sides to get $yx^2 = \int x^4dx = \frac{x^5}{5} + c \Rightarrow y = \frac{\frac{x^5}{5}+c}{x^2} = \frac{x^3}{5} + \frac{c}{x^2}$.

4. To write the equation in the form $y' + Py = Q$, you need to divide by x^2 first. Obtain $y' + \frac{1}{x}y = \frac{1}{x^2}$. This gives you that $P = \frac{1}{x}$. Determine the integrating factor now. $I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. Multiply the equation by x to get $y'x + y = \frac{1}{x}$. Note that the left side is the derivative of the product yx. So, the equation becomes $(yx)' = \frac{1}{x}$. Integrate both sides to get $yx = \int \frac{1}{x}dx = \ln x + c \Rightarrow y = \frac{\ln x + c}{x}$.

   Using the initial condition $y(1) = 2$, you have $2 = \frac{0+c}{1} \Rightarrow c = 2$. Thus the solution is $y = \frac{\ln x + 2}{x}$.

5. Divide by x first to get $y' + \frac{2}{x}y = \frac{\cos x}{x}$. $P = \frac{2}{x} \Rightarrow I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$. Multiply by I to get $y'x^2 + 2xy = x\cos x \Rightarrow (yx^2)' = x\cos x \Rightarrow yx^2 = \int x\cos x dx$. Using the integration by parts with $u = x$ and $dv = \cos x dx$, obtain that $yx^2 = x\sin x + \cos x + c$. Divide by x^2 to get the general solution $y = \frac{x\sin x + \cos x + c}{x^2} = \frac{1}{x}\sin x + \frac{1}{x^2}\cos x + \frac{c}{x^2}$.

   With $y(\pi) = 0$ you have that $0 = \frac{-1}{\pi^2} + \frac{c}{\pi^2} \Rightarrow 0 = -1 + c \Rightarrow c = 1$. Thus, the particular solution is $y = \frac{1}{x}\sin x + \frac{1}{x^2}\cos x + \frac{1}{x^2}$.