Matlab Notes for Calculus 2

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Content

1. Algebra and Functions

1.1 Basic Arithmetic

You can use +, -, *, \ and ^ to add, subtract, multiply, divide or exponentiate, respectively. For example if you enter:

>> 2^3 - 2*2
Matlab calculates the answer:
ans =   4

If you want to perform further calculations with the value of the answer, you can type **ans** rather than retyping the specific answer value. For example,
>> sqrt(ans)
ans =   2

To perform symbolic calculations in Matlab, use **syms** to declare the variables you plan to use. For example, suppose that you need factor x^2-3x+2. First you need
>> syms x   (you are declaring that x is a variable)

Then you can use the command **factor**.
>> factor(x^2-3*x+2)
ans = (x-1)*(x-2)

Note that we entered **3*x** to represent $3x$ in the command above. **Entering * for multiplication is always necessary in Matlab.**

Besides **factor** command, you have **simplify** and **expand**.

Practice problems 1

1. Factor $x^3+3x^2y+3xy^2+y^3$.   2. Simplify $\dfrac{x^3-8}{x-2}$.

3. Evaluate the following expressions a) $\sin(\pi/6)$ b) $\dfrac{\sqrt{5}+3}{\sqrt{3}-1}$ c) $\log_2(5)$

Solutions
1. **syms x y** followed by **factor(x^3+3*x^2*y+3*x*y^2+y^3)** gives you **ans=(x+y)^3**
2. **syms x** followed by **simplify((x^3-8)/(x-2))** gives you **ans=x^2+2x+4**
3. Enter **sin(pi/6)**. Then **ans=.5** b) Enter **(sqrt(5)+3)/(sqrt(3)-1)**. Then **ans=7.152** c) Enter **log(5)/log(2)**. Then **ans=2.3219**.

1.2 Solving Equations

For solving equations, you can use the command **solve**. The command **solve** is always followed by parenthesis. After that, the equation you would like to solve should be entered in single quotes. Separated by a coma, the equation is followed by the variable for which you are solving the equation in (single) quotes. Thus, the command **solve** has the following form

solve('_equation_**', '**_variable for which you are solving_**')**

For example, to solve the equation $x^3-2x-4=0$, you type:
>> solve('x^3-2*x-4=0')
and get the following answer:
ans = [2]   [-1+i]   [-1-i]
Here i stands for the imaginary number $\sqrt{-1}$. This answer tells us that there is just one real solution, 2.

Matlab can give you both symbolic and numerical answer. For example, let us solve the equation $3x^2-8x+2=0$.

>> solve('3*x^2-8*x+2=0','x')
ans =   [4/3+1/3*10^(1/2)]   [4/3-1/3*10^(1/2)]

If we want to get the answer in the decimal form with, say, three significant digits, we can use the command **vpa**.
>> vpa(ans, 3)
ans =   [2.38]   [0.28]

The command **vpa** has the general form
          vpa(*expression you want to approximate***, *****number of significant digits***)**

You can solve more than one equation simultaneously. For example suppose that we need to solve the system $x^2+ x+ y^2 = 2$ and $2x-y = 2$. We can use:
>> [x,y] =solve('x^2+ x+ y^2 = 2', '2*x-y = 2')
to get the answer
x =   [2/5]   [1]
y =   [-6/5]   [0]

Note that the **[x,y]=** part at the beginning of the command was necessary since without it Matlab produces the answer:
ans =   x: [2x1 sym]   y: [2x1 sym]
This answer tells us just that the solutions are two values of the pair (x,y) but we do not get the solutions themselves. To get the solution vectors displayed, we must use **[x,y]=** before the command **solve**.

You can solve an equation in two variables for one of them. For example the command
>> solve('y^2-5*x*y-y+6*x^2+x=2', 'y')
solves the given equation for values of y in terms of x. The answer is:
ans =   [3*x+2]   [2*x-1]

Practice problems 2
1. Solve the following equations and express the answers as decimal numbers.
          a) $x^3-2x+5=0$ b) $\log_2(x^2-9)=4$.
2. Solve $5x+2y+4z = 8,\ -3x+y+2z = -7,\ 2x+y+z = 3$ for $x,\ y$ and z.

Solutions. 1. a) Enter **solve('x^3-2*x+5=0', 'x').** The only real answer is **-2.09.**
b) Enter **solve('log(x^2-9)/log(2)=4','x').** The answers are **ans= 5, -5.**
2. **[x,y,z]=solve('5*x+2*y+4*z = 8', '-3*x+y+2*z = -7', '2*x+y+z = 3') x=2 y=-1 z=0**

          1.3 Representing a function

The following table gives an overview of how most commonly used functions or expressions are represented in Matlab.
To represent a function, use the command **inline**. Similarly to **solve**, this command is followed by parenthesis and has the following form:

function or symbol	representation in MATLAB
e^x	exp(x)
ln x	log(x)
log x	log(x)/log(10)
log. base a of x	log(x)/log(a)
sin x	sin(x)
cos x	cos(x)
arctan(x)	atan(x)
π	pi

inline('*function*', '*independent variable of the function*')

Here is how to define the function x^2+3x-2:
>> f = inline('x^2+3*x-2', 'x')
f =
  Inline function:
  f(x) = x^2+3*x-2

After defining a function, we can evaluate it at a point. For example,
>> f(2)
ans =   8

In some cases, we will need to define function f as a vector. Then we use:
>> f = inline(vectorize('x^2+3*x-2'), 'x')
f =
  Inline function:
  f(x) = x.^2+3.*x-2

In this case, we can evaluate a function at more than one point at the same time. For example, to evaluate the above function at 1, 3 and 5 we have:
>> f([1 3 5])
ans =   2   16   38

If a function is short, it might be **faster to evaluate a function at a point simply by typing the value of *x* directly for *x*.** For example, to evaluate $\sin(x)$ at $x=2$, simply type
>> sin(2)
and obtain the answer
ans = .909297

As when using the calculator, one must be careful when representing a function. For example
- $\dfrac{1}{x(x+6)}$ should be represented as **1/(x*(x+6))** not as **1/x*(x+6)** nor as **1/x(x+6),**
- $\dfrac{3}{x^2+5x+6}$ should be represented as **3/(x^2+5*x+6)** not as **3/x^2+5*x+6,**
- e^{5x^2} should be represented as **exp(5*x^2)** not as **e^(5*x^2), exp*(5*x^2), exp(5x^2)** nor as **exp^(5*x^2).**
- $\text{Ln}(x)$ should be represented as **log(x)**, not **ln(x).**
- $\log_3(x^2)$ should be represented as **log(x^2)/log(3)** not as **log(x)/log(3)*x^2.**

2. Graphics

Let us start by declaring that *x* is a variable:
>> syms x

The simplest command in Matlab for graphing is **ezplot**. The command has the following form

ezplot(*function*)

For example, to graph the function x^2+x+1, you simply type
>> ezplot(x^2+x+1)

A new window will open and graph will be displayed. To copy the figure to a text file, go to **Edit** and choose **Copy Figure**. Then place cursor to the place in the word file where you want the figure to be pasted and choose **Edit** and **Paste**.

We can specify the different scale on *x* and *y* axis.
To do this, the command **axis** is used.
It has the following form

$$\textbf{axis([}x_{min}, \ x_{max}, \ y_{min}, \ y_{max}\textbf{])}$$

This command parallels the commands in menu WINDOW on the TI83 calculators.

For example, to see the above graph between x-values -10 and 10 and y-values 0 and 60, you can enter
>> axis([-10 10 0 60])

Note that the domain of function did not change by command axis. To see the graph on the entire domain (in this case [-10, 10]), add that domain after the function in the command ezplot:

$$\textbf{ezplot(}function\textbf{, [}x_{min}, \ x_{max}\textbf{])}$$

In this case,
>> ezplot(x^2+x+1, [-10, 10])
will give you the desired graph.

For the alternative command for graphics, **plot,** you can find more details by typing **help**.

x^2+x+1

x^2+x+1

x^2+x+1

To graph multiple curves on the same plot, you can also use the **ezplot** command. For example to graph the functions $\sin(x)$ and e^{-x^2}, you can use:
>> ezplot(sin(x))
>> hold on
>> ezplot(exp(-x^2))
>> hold off

Practice problems 3

1. Let f(x)= $\dfrac{x^3+x+1}{x}$
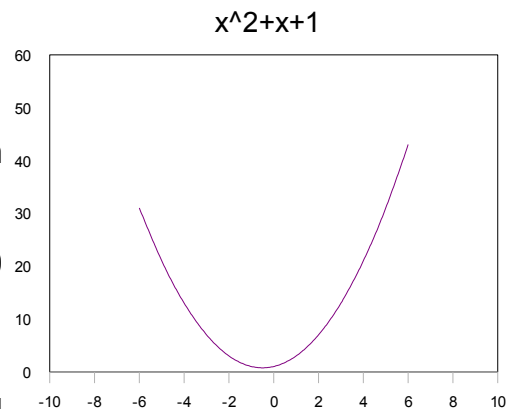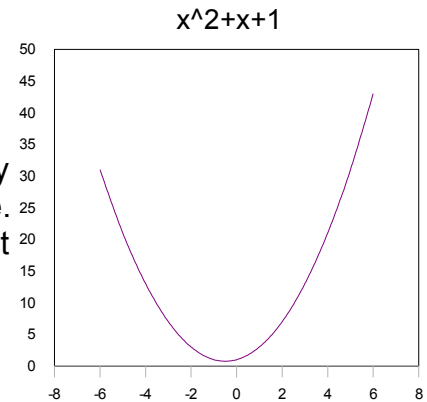
a) Represent f(x) as a function in Matlab. Then evaluate it at x=3 and x=-2.
b) Find x-value(s) that corresponds to y-value y=2.
c) Graph f(x) on domain [-4 4].

2. Graph ln(x+1) and 1-x² on the same plot for x in [-2 6] and y in [-4 4].

Solutions
1. a) **>> f=inline('(x^3+x+1)/x', 'x')**,   **>> f(3)   ans= 10.333,   >>f(-2)   ans=4.5.**
b) The problem is asking you to solve equation $\dfrac{x^3+x+1}{x}$ =2. Using solve command,
solve('(x^3+x+1)/x=2','x'). you get one real answer x=-1.3247
c) **ezplot((x^3+x+1)/x, [-4,4])**.

3. Solving Equations in Matlab using fzero

In some cases, the command **solve** may fail to produce all the solutions of an equation. In those cases, you can try to find solutions using **fzero** (short for "find zero") command. In order to use the command, first you need to write equation in the form

$$f(x)=0.$$

Thus, put all the terms of th equations on one side leaving just zero on the other. To find a solution near the x-value x=a, you can use

fzero('*left side of the equation*', a)

The command **fzero**, similarly as **solve** is always followed by expression in parenthesis. The equation should be in single quotes.

If it is not clear what a convenient x-value a should be, you may want to graph the function on the left side of the equation first, check where it intersects the x-axis. Alternatively, you can graph left and right side of the equation that is not in f(x)=0 form and see where the two functions intersect. Then decide which x-value you should use.

Example. To solve the equation $e^{x^2}-2=x+4$, we can first graph the functions on the left and right side of the equation using
    syms x   ezplot(exp(x^2)-2)   hold on   ezplot(x+4)   hold off

From the graph, we can see that the two functions intersect at a value near -1 and at a value near 1. To use **fzero**, we need to represent the equation in the form $e^{x^2}-2-(x+4)=0$ (or simplified form $e^{x^2}-x-6=0$). Then, we can find the positive solution by using **fzero** to find a zero near 1 and then to find the negative solution near -1, for example. Thus, both solutions can be obtained by:
>> fzero('exp(x^2)-2-(x+4)', 1)   ans = 1.415
>> fzero('exp(x^2)-2-(x+4)', -1)   ans = -1.248

Note also that the command **solve('exp(x^2)-2=x+4', 'x')** returns just the positive solution. Thus, knowing how to use **fzero** command may be really useful in some cases.

4. Calculus

4.1 Differentiation

Start by declaring x for a variable. The command for differentiation is **diff**. It has the following form

$$\text{\textbf{diff}}(function)$$

For example,
>> syms x
>> diff(x^3-2*x+5)
gives us the answer **ans = 3*x^2-2**

To get n-th derivative use

$$\text{\textbf{diff}}(function, n)$$

For example, to get the second derivative of x^3-2x+5, use: **>> diff(x^3-2*x+5, 2)**
ans = 6*x
Similarly, the 23rd derivative of sin(x) is obtained as follows.   **>> diff(sin(x), 23)**
ans =-cos(x)

To evaluate derivative at a point, we need to represent the derivative as a new function. For example, to find the slope of a tangent line to x^2+3x-2 at point 2, we need to find the derivative and to evaluate it at $x=2$.
>> diff(x^2+3*x-2) (first we find the derivative)   **ans = 2*x+3**
>> f = inline('2*x+3', 'x') (then we representative the derivative as a function)
f =   Inline function: f(x) = 2*x+3
>> f(2) (and, finally, we evaluate the derivative at 2)
ans =   7

Practice problems 4

1. Let f(x)= $\dfrac{x^3+x+1}{x}$ a) Use Matlab to find the first derivative of f(x). b) Evaluate the first derivative at x=1.
2. Let f(x)=e $^{3x^2+1}$. a) Find the first derivative of f(x). b) Find the slope of the tangent line to f(x) at x=1. c) c) Find the critical points of f(x).
3. Find the 12th derivative of the function $(\frac{x}{2}+1)^{65}$.

Solutions.
1.   a) **syms x   diff((x^3+x+1)/x)**   **ans = 2*x-1/x^2   or (2*x^3-1)/x^2.**
          b) Inline the derivative: **g=inline('2*x-1/x^2','x')**. Then **g(1)** gives you **ans=1**.

2.   a) **diff(exp(3*x^2+1))**   **ans=6*x*exp(3*x^2+1)**
        b) Represent the derivative as function: **g=inline('6*x*exp(3*x^2+1)','x').** Then
        evaluate **g(1)**. Get 6*exp(4). To see the answer as a decimal number (say to five
        nonzero digits) use vpa(ans, 5). Get 327.58.
        c) **solve('6*x*exp(3*x^2+1)=0','x')**   **ans=0**
3. **diff((x/2+1)^65, 12)**

        4.2 Integration

We can use Matlab for computing both definite and indefinite integrals using the command
int. For the indefinite integrals, start with **syms x** followed by the command

$$int(function)$$

For example, the command
>> int(x^2)

evaluates the integral $\int x^2 dx$ and gives us the answer   **ans = 1/3*x^3**

For definitive integrals, the command is

$$int(function, lower bound, upper bound)$$

For example,
>> int(x^2, 0, 1)

evaluates the integral $\int_0^1 x^2 dx$ The answer is   **ans = 1/3**

In Matlab, **Inf** stands for positive infinity. Matlab can evaluate the (convergent) improper
integrals as well. For example:
 >> int(1/x^2, 1, Inf)
ans = 1

For the divergent integrals, Matlab gives us the answer infinity. For example:
>> int(1/x, 0, 1)
ans = inf

Matlab can evaluate the definitive integrals of the functions that do not have elementary
primitive functions. Recall that the integrals $\int \frac{\sin x}{x} dx, \int \frac{e^x}{x} dx, \int e^{x^2} dx$

can not be represented via elementary functions. Suppose that we need to find the integral of
$\frac{\sin x}{x}$ from 1 to 3. The command **>> int(sin(x)/x, 1, 3)**

doesn't gives us a numerical value. We have just: **ans =**   **sinint(3)-sinint(1)**
Using the command **vpa,** we obtain the answer in numerical form. For example,
>> vpa(ans, 4) gives us   **ans =   0.9026**

Practice problems 5

1. Evaluate the following integrals.   a) $\int xe^{-3x}dx$ b) $\int_0^1 xe^{-3x}dx$.

2. Using Matlab, determine if the following integrals converge or diverge. If they converge, evaluate them.

a) $\int_1^\infty \frac{3}{x^2+5x+6}dx$

b) $\int_1^2 \frac{1}{(x-1)(x+1)}dx$

Solutions.

1.   a) **syms x**   **int(x*exp(-3*x))**   **ans=-1/3*x*exp(-3*x)-1/9*exp(-3*x)**
        b) **int(x*exp(-3*x), 0,1)**   **ans=-4/9*exp(-3)+1/9**   **vpa(ans, 4)**   **ans=.08898**
2. a) **int(3/(x^2+5*x+6), 1, Inf)**   **vpa(ans, 4)**   **ans= .8338** So, the integral is convergent.
        b) **int(1/((x-1)*(x+1)), 1, 2)**   **ans= Inf** So, the integral is divergent.

        4.3 Limits

You can use **limit** to compute limits, left and right limits as well as infinite limits. For example, to evaluate the limit when $x \to 2$ of the function $\frac{x^2-4}{x-2}$, we have:

>> syms x
>> limit((x^2-4)/(x-2), x, 2)   **ans = 4**

You can also evaluate left and right limits. For example:
>> limit(abs(x)/x, x, 0, 'left')   **ans = -1**
>> limit(abs(x)/x, x, 0, 'right')   **ans = 1**

Limits at infinity:
>> limit(exp(-x^2-5)+3, x, Inf)   **ans = 3**

Practice problem 6

Find the limits of the following functions at indicated values.

a) $f(x)=\frac{x^{12}-1}{x^3-1}$, $x \to 1$ b) $f(x)= 3+e^{-2x}$, $x \to \infty$ c) $f(x)=\frac{6x^3-4x+5}{2x^3-1}$, $x \to \infty$

Solutions.
a) syms x   limit((x^12-1)/(x^3-1), x, 1)   ans=4
b) limit(3+exp(-2*x), x, Inf)   ans=3
c) limit((6*x^3-4*x+5)/(2*x^3-1), x, Inf)   ans=3

5. Graphic Continued

        5.1 Parametric Plots

We can use the command **ezplot** to graph a parametric curve as well. For example, to graph a circle $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$, we have:
>> ezplot('cos(t)', 'sin(t)', [0, 2*pi])

5.2 Special effects

You can change the title above the graph by using the command **title**. For example,
>> ezplot('x^2+2*x+1')
>> title 'a parabola'
You can add labels to x and y axis by typing **xlabel** and **ylabel.**

You can add text to your picture. For example, suppose that we want to add a little arrow pointing to the minimum of this function, the point (-1, 0). We can do that with the command:
>> text(-1, 0, '(-1, 0) \leftarrow a minimum')

You can produce an animated picture with **comet**. This command produces a parametric plot of a curve just as **ezplot** does, except that you can see the curve being traced out in time. For example, we can trace the motion on the circle $x = \cos t$, $y = \sin t$ by using
>> t = 0:0.1:4*pi;   (meaning that t has values between 0 and 4π, 0.1 step away from each other)
>> comet(cos(t), sin(t))

If the point is moving too fast, you can reparameterize the same circle as follows
>> t = 0:0.1:200*pi;
>> comet(cos(t/50), sin(t/50))

Practice problems 7
1. Graph the parametric curve $x = t \cos t$, $y = t \sin t$ for $0 \le t \le 10\pi$.
2. Trace the curve $x = t \cos (t/20)$, $y = t \sin (t/20)$ in time for $0 \le t \le 200\pi$.

Solutions
1. **syms t   ezplot(t*cos(t), t*sin(t), [0, 10*pi])**   **2. t = 0:0.5:200*pi;**
comet(t*cos(t/20), t*sin(t/20))

6. Differential Equations

6.1 Basics of Differential Equations

We can use Matlab to solve differential equations. The command for finding the symbolic solution is **dsolve**. For that command, the derivative of the function y is represented by **Dy**. The command has the following form:

dsolve('*equation*', '*independent variable*')

For example, suppose that we want to find the **general solution** of the equation $x y' - y = 1$.

You can do that by:
 >> dsolve('x*Dy-y=1', 'x')
ans = -1+x*C1
This means that the solution is any function of the form $y = -1 + cx$, where c is any constant.

If we have the initial condition, we can get the **particular solution** on the following way:

dsolve('*equation*', '*initial condition*', '*independent variable*')

For example, to solve $x\,y' - y = 1$ with the initial condition $y(1)=5$, we can use:
>> dsolve('x*Dy-y=1', 'y(1)=5', 'x')
ans = -1+6*x
To graph this solution, we can simply type:
>> ezplot(ans)

We can graph a couple of different solutions on the same chart by using **hold on** and **hold off** commands. For example, to graph the solutions of differential equation y'=0.1y(1-y) for several different initial conditions, y(0)=0.1, y(0)=0.3 y(0)=0.5 and y(0)=0.7, first find the four particular solutions, then graph them using **hold on** and **hold off** commands. To observe the limiting behavior of the solution, in examples below the interval [0, 100] is used as the domain.

Finding the four solutions:
>> dsolve('Dy=0.1*y*(1-y)','y(0)=0.1', 'x')
ans = 1/(1+9*exp(-1/10*x))
>> dsolve('Dy=0.1*y*(1-y)','y(0)=0.3' 'x')
ans = 1/(1+7/3*exp(-1/10*x))
>> dsolve('Dy=0.1*y*(1-y)','y(0)=0.5' 'x')
ans = 1/(1+exp(-1/10*x))
>> dsolve('Dy=0.1*y*(1-y)','y(0)=0.7' 'x')
ans = 1/(1+3/7*exp(-1/10*x))
Graphing the fours solutions:
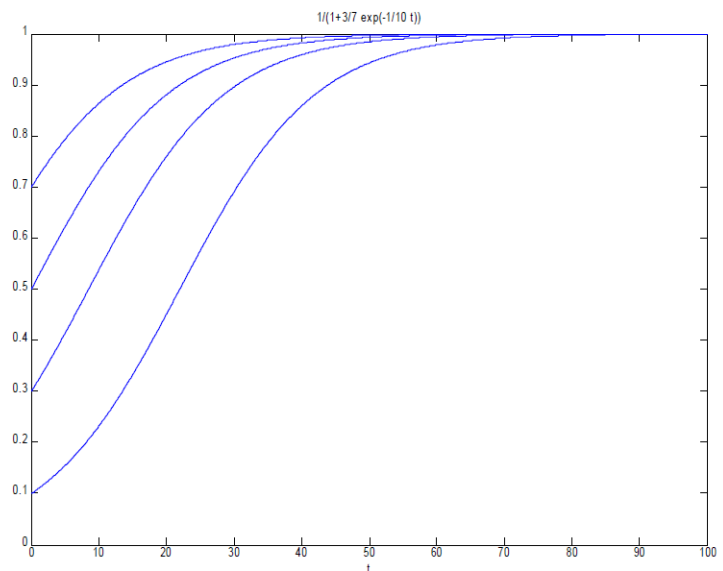**>> ezplot('1/(1+9*exp(-1/10*x))',[0,100])**
>> hold on
>> ezplot('1/(1+7/3*exp(-1/10*x))',[0,100])
>> ezplot('1/(1+exp(-1/10*x))',[0,100])
>> ezplot('1/(1+3/7*exp(-1/10*x))',[0,100])
>> hold off
>>axis([0,100,0,1])

6.2 Second Order Equations

The second order linear equations can be solved similarly as the first order differential equations by using **dsolve** or **ode45**. For the command **dsolve**, recall that we represent the first derivative of the function y with **Dy**. The second derivative of y is represented with **D2y**. For example, the command for solving $y''-3y'+2y = \sin x$.

>> dsolve('D2y-3*Dy+2*y=sin(x)', 'x')
ans = 3/10*cos(x)+1/10*sin(x)+C1*exp(x)+C2*exp(2*x)

If we have the initial conditions y(0) = 10, y'(0)=-10, we would have:
>> dsolve('D2y-3*Dy+2*y=sin(x)', 'y(0)=1', 'Dy(0)=-1', 'x')
ans = 3/10*cos(x)+1/10*sin(x)+5/2*exp(x)-9/5*exp(2*x)

Practice problems 8
1. a) Find the general solution of the differential equation $y'-2y=6x$.
   b) Find the particular solution with initial condition y(0)=3.
   c) Plot the particular solution on interval [0,2] and find the value of this solution at 2.
2. Graph the solutions of the differential equation $y' = x+y$ for the y-values of the initial condition y(0) taking integer values between -2 and 4.
3. Find the general solution of the equation $y''-4 y'+4 y= e^x +x^2$. Then find the particular solution with the initial values y(0)=8, y'(0)=3.

Solutions.
1. a) General solution: **dsolve('Dy-2*y=6*x', 'x')**   ans= (-6x-3)/2+C1*exp(2x)
b) Paticular solution: **dsolve('Dy-2*y=6*x', 'y(0)=3', 'x')**   ans= (-6x-3+9*exp(2x))/2
c) **syms x   ezplot((-6x-3+9*exp(2x))/2, [0 2])** To find the value at 2: **f=inline((-6x-3+9*exp(2x))/2, x)**   **f(2)**   ans=238.19
2. First, find the seven particular solutions of $y' = x+y$ with initial conditions y(0)=-2,-1,0,1,2,3,4.
s1 = dsolve('Dy = x+y', 'y(0)=-2', 'x')
s2 = dsolve('Dy = x+y', 'y(0)=-1', 'x') ... etc ..
s7 = dsolve('Dy = x+y', 'y(0)=4', 'x')
Then plot them on the same graph.
hold on
ezplot(s1)
ezplot(s2) ...etc...
ezplot(s7)
hold off
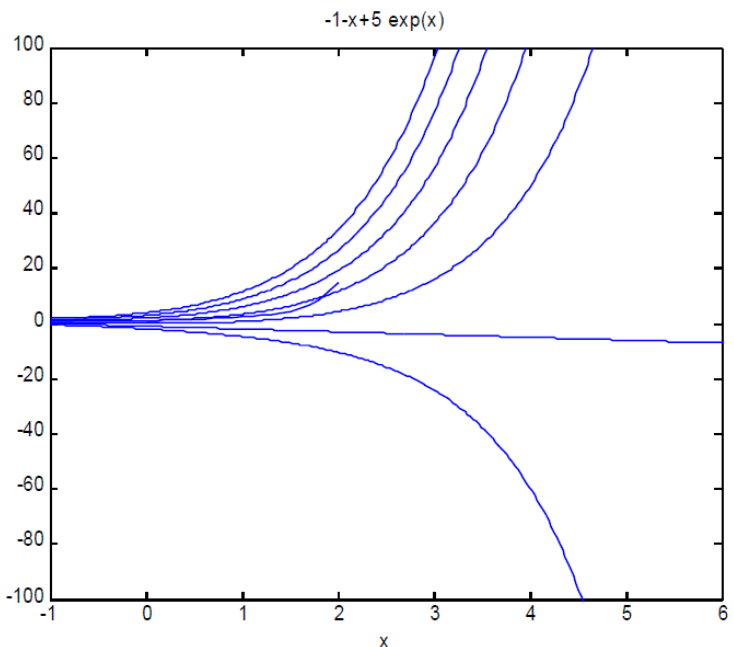
3. **>> dsolve('D2y-4*Dy+4*y=exp(x)+x^2', 'x')**
**ans = exp(x)
+1/4*x^2+1/2*x+3/8+C1*exp(2*x)
+C2*exp(2*x)*x**
>> dsolve('D2y-4*Dy+4*y=exp(x)+x^2','y(0)=8', 'Dy(0)=3', 'x')
**ans   =   exp(x)
+1/4*x^2+1/2*x+3/8+53/8*exp(2*x)-
47/4*exp(2*x)*x**

-1-x+5 exp(x)