Calculus 2
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Modeling with Differential Equations

In application problems or real life scenarios, one may need to come up with a differential equation that accurately describes certain scenario *first,* before solving it. So, being able to model the problem using an equation is equally important as being able to solve the equation. The process of writing an equation describing the given situation is referred to as *mathematical modeling.* In order to successfully model a problem by a differential equation, it might be helpful to ask yourself the questions listed below.

1. **Identify the real problem. Identify the problem variables.** What do we need to describe or find out? What is the problem asking for?

2. **Construct an appropriate relation between the variables – a differential equation.** Determine how the dependent variable, the independent variable and the rate of change are connected. Figuring this out results in a differential equation that models the problem.

3. **Obtain a mathematical solution.** Recognize the type of the equation. Decide if you can solve it analytically ("by hand") or if you need to find a numerical solution using technology. In the first case, decide on the method that you will use (in this course, decide whether the equation is separable or linear).

4. **Interpret the mathematical solution.** After solving the equation, check if the mathematical answer agrees with the context of the original problem. **Check the validity**: Does your answer make sense? Do the predictions agree with real data? Do the values have the correct sign? Correct units? Correct size?

Example 1. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria. Find the number of bacteria after 4 hours.

Solution. Identifying variables: let y stands for the bacteria culture and t stands for time passed. The first part of the problem "A bacteria culture starts with 500 bacteria..." tells us that $y(0) = 500$. The second part "... and grows at a rate proportional to its size" is the key for getting the mathematical model. Recall that the rate is the derivative and that "...is proportional to.." corresponds to "...is equal to a constant multiple of..." So, the equation relating the variables is $\frac{dy}{dt} = ky$. The solution of this differential equation is $y = y_0 e^{kt}$. Since $y_0 = 500$, it remains to determine the proportionality constant k. From the condition "After 3 hours there are 8000 bacteria" we obtain that $8000 = 500e^{3k}$ which gives us that $k = \frac{1}{3}\ln 16 = .924$. Thus, the number of bacteria after t hours can be described by $y = 500e^{.924t}$.

Using the function we have obtained, we find the number of bacteria after 4 hours to be $y(4) =$ 20159 bacteria.

Another often encountered scenario is when the total rate $\frac{dy}{dt}$ is computed as the difference of the rate causing an increase and the rate causing a decrease

$$\mathbf{\frac{dy}{dt} = rate\ in - rate\ out.}$$

The following example illustrates this situation.

Example 2. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the concentration at that time.

1. Set up the differential equation that models this situation.

2. If $r = 4$ and the proportionality constant is 2, sketch the graphs of general solutions and examine their stability. Determine the concentration of the glucose after a long period of time.

3. Suppose that the initial concentration is 1 mg/cm^3. Solve the equation with this initial condition and sketch the graph of this solution.

Solution. (a) Let y denotes the concentration at time t. The concentration is increasing at a constant rate r and is decreasing at a rate proportional to y. Let k denote the proportionality constant. Then the rate in is r and the rate out is ky. So, the differential equation $y' = r - ky$ models this situation.

(b) With the given values, the equation becomes $y' = 4 - 2y$. The equilibrium solution is $4 - 2y = 0 \Rightarrow 4 = 2y \Rightarrow y = 2$. $y' = 4 - 2y > 0$ for $y < 2$ and $y' = 4 - 2y < 0$ for $y > 2$. Thus, the solutions are decreasing toward the equilibrium solution if $y_0 > 2$ and increasing towards 2 if $y_0 < 2$. So, $y = 2$ is stable. This means that regardless of the initial concentration, the concentration will become 2 mg/cm^3 after sufficiently long time period.

(c) Separate the variables $\frac{dy}{dt} = 4 - 2y \Rightarrow \frac{dy}{4-2y} = dt$. Integrate both sides. Use $u = 4 - 2y$ for the left side. Obtain $\frac{1}{-2}\ln(4-2y) = t + c \Rightarrow \ln(4-2y) = -2t - 2c \Rightarrow 4 - 2y = e^{-2t-2c} \Rightarrow 2y = 4 - e^{-2t-2c} \Rightarrow y = 2 - \frac{1}{2}e^{-2t-2c} = 2 - \frac{e^{-2c}}{2}e^{-2t}$. Denoting $\frac{e^{-2c}}{2}$ by C, we obtain that $y = 2 - Ce^{-2t}$. Note that the term Ce^{-2t} converges to 0 and so $y \to 2$ for $t \to \infty$ regardless of the value of the constant C. This agrees with the conclusion from part b).

If $y(0) = 1$, then $1 = 2 - Ce^0 \Rightarrow 1 = 2 - C \Rightarrow C = 1$. Thus $y = 2 - e^{-2t}$.

Practice Problems.

1. The rate of change of the velocity of a raindrop is proportional to the difference of the velocity and a constant. If the initial velocity is v_0, set up the differential equation and the initial condition for finding the velocity $v(t)$ of a raindrop at the time t. Do not solve the equation.

2. A population of bacteria grows at a rate proportional to the size of population. The proportionality constant is 0.7. Initially, the population consist of two members. Find the population size after six days.

3. Experiments show that if the chemical reaction $N_2O_5 \to 2NO_2 + \frac{1}{2}O_2$ takes place at 45 degrees Celsius, the rate of reaction of dinitrogen pentoxide is proportional to its concentration $C(t)$ with proportionality constant equal to -.0005.

   (a) Find an expression for the concentration of dinitrogen pentoxide if the initial concentration is C_0.

   (b) Determine how long it takes for the concentration of N_2O_5 to be reduced to 90% of its original value.

4. Let $A(t)$ be the area of tissue culture at time t (in days). Let the final area of the tissue when the growth is complete be 10 cm^2. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to $\sqrt{A}$. So, a reasonable model for the growth of tissue is obtained by assuming that the rate of growth is proportional to the product of $\sqrt{A}$ and $10 - A$. Formulate a differential equation that models this situation.

5. Let a be a positive number. Suppose that y is a function whose derivative is proportional to y^{1+a}. Differential equation that models this situation is

$$y' = ky^{1+a}$$

This differential equation is called the **doomsday equation** because the exponent of y on the right side of the equation is larger than that for the natural growth ($1 + a > 1$). This produces a solution with a vertical asymptote. The x-value corresponding to this asymptote is called **the doomsday** because y-values diverge to infinity already when the x-values approach this finite value.

The size of an especially prolific breed of rabbits can be modeled by this differential equation. Suppose that $k = 1$, $a = \frac{1}{100}$ there are 2 rabbits initially, and the time is measured in months. Determine the doomsday i.e. the time when there will be infinitely many rabbits.

Solutions.

1. The initial value problem is $\frac{dv}{dt} = k(v - c), \quad v(0) = v_0$.

2. The equation is $\frac{dy}{dt} = 0.7y$. Separating the variables and solving gives you $y = Ce^{0.7t}$. Using the initial condition $y(0) = 2$, the solution becomes $y = 2e^{0.7t}$. Plugging 6 for t produces $y(6) = 133$ protozoa.

3. (a) The equation is $\frac{dC}{dt} = -.0005C$. Separating the variables and solving gives you $C = C_0e^{-.0005t}$. (b) Solve $0.9C_0 = C_0e^{-.0005t}$ for t. Get $t = \frac{\ln 0.9}{-.0005} = 210.72$ seconds or about 3.5 minutes.

4. $\frac{dA}{dt} = k\sqrt{A}(10 - A)$

5. $\frac{dy}{dt} = y^{1.01} \Rightarrow y^{-1.01}dy = dt \Rightarrow \frac{y^{-0.01}}{-0.01} = t + c \Rightarrow y^{-0.01} = -.01t - .01c = -0.01t + C \Rightarrow y = (-0.01t + C)^{-100}$. Using the initial condition $y(0) = 2$, obtain that $2 = C^{-100} \Rightarrow C = 2^{-.01} = .993$. So, the particular solution is $y(t) = (-.01t+0.993)^{-100} = \frac{1}{(-.01t+0.993)^{100}}$. This function has a vertical asymptote at x-value corresponding to a zero of the denominator. $-.01t + 0.993 = 0 \Rightarrow t = \frac{0.993}{0.01} = 99.3$. So, the doomsday is in 99.3 months.