Calculus 2
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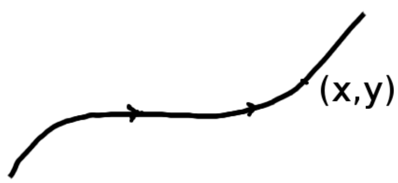
Parametric Curves

In the past, we mostly worked with curves in the form $y = f(x)$. However, this format does not encompass all the curves one encounters in applications. For example, consider the circle $x^2+y^2 = a^2$. Solving for y does not give you one but two functions $y = \pm\sqrt{a^2 - x^2}$ and the implicit equation $x^2 + y^2 = a^2$ may not be the appropriate format in many cases. This example indicates the need for another approach to representations of curves.

Assume that the variables x and y are given as functions of a new parameter t as



$$x = x(t) \text{ and } y = y(t)$$

In this case the points $(x, y) = (x(t), y(t))$ constitute the graph of a **parametric curve**.

Careful: don't think of the equations $x = x(t)$ and $y = y(t)$ as the equations of two object as they represent a *single* curve: the first equation describes the x-coordinate and the second the y-coordinate on the given curve.

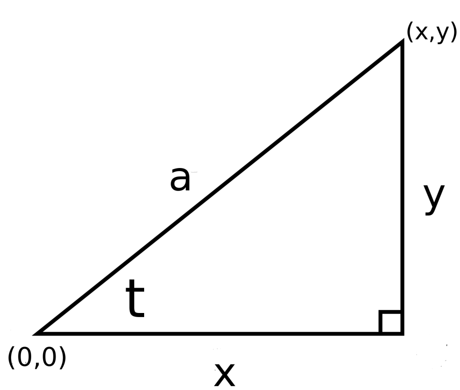
If $a \leq t \leq b$, the points $(x(a), y(a))$ and $(x(b), y(b))$ on the parametric curve $x = x(t),\ \ y = y(t)$ are called the **initial** and the **terminal point** respectively.

The following are some of the advantages of this approach.

1. It provides a good physical interpretation: the position (x, y) depends on the time t. It also allows a direct generalization to three dimensions when position of point (x, y, z) depends on time t (more about this in Calculus 3).

2. It enables one to consider **orientation** of a curve i.e. the direction of movement when t increases. One can also reparametrize a curve to change the orientation.

3. Some important cases of implicit curves can be represented parametrically.



For example, circle $x^2 + y^2 = a^2$ can be parametrized by

$$x = a\cos t \quad \text{and} \quad y = a\sin t.$$

In this parametrization t corresponds to the angle between the position vector of the point (x, y) and the positive part of x-axis. Note that $\cos t = \frac{x}{a} \Rightarrow a\cos t = x$ and $\sin t = \frac{y}{a} \Rightarrow a\sin t = y$.

Also note that the parametric equations $x = a\cos t$ and $y = a\sin t$ satisfy the implicit equation $x^2 + y^2 = a^2$ since $(a\cos t)^2 + (a\sin t^2) = a^2\cos^2 t + a^2\sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2$.

The circle of radius a centered at (x_0, y_0) can be parametrized as

$$x = x_0 + a\cos t \text{ and } y = y_0 + a\sin t \text{ with } 0 \le t \le 2\pi.$$

Note that these equations satisfy the implicit equation $(x - x_0)^2 + (y - y_0)^2 = a^2$.

To graph a parametric curve on your calculator, go to **Mode** and switch from **Func** to **Par**. This will switch your calculator to the parametric mode. In this mode, you can enter both x and y equations when pressing **Y=** key. Use key **X,T,θ,n** to display the variable t when needed.

Note also that the standard window on your calculator is set to be $0 \le t \le 2\pi$. So, in cases when you want to see the graph for negative t values you have to manually edit the window (**ZOOM Standard** will give you the same standard t-interval $[0, 2\pi]$). The command **ZOOM Fit** may display window set to note the limiting behavior and that make some other features of the curve (e.g. a loop) not visible.

The derivative of a parametric curve. The slope of the tangent to the parametric curve $x = x(t)$, $y = y(t)$ represents the rate of change $\frac{dy}{dx}$ at a point. This rate can be computed as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

The **second derivative** can be obtained by differentiating the first derivative as follows

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

To find the line tangent to the curve $x = x(t)$, $y = y(t)$ at $t = t_0$ note that this line passes the point $(x(t_0), y(t_0))$ and has the slope $m = \frac{y'(t_0)}{x'(t_0)}$.

Recall that a horizontal tangent corresponds to $\frac{dy}{dx} = 0 \Rightarrow dy = y'(t)dt = 0$ and the vertical tangent corresponds to $\frac{dy}{dx}$ not being defined. In most cases you will be able to find it by setting the denominator dx equal to zero. Thus $dx = x'(t)dt = 0$.

The area enclosed by a parametric curve. To compute the area enclosed by the parametric curve $x = x(t)$, $y = y(t)$ on interval $t \in [t_1, t_2]$ is

$$A = \int_a^b y \;\; dx = \pm \int_{t_1}^{t_2} y \;\; x' \, dt$$

The sign may be negative depending on the orientation. If the curve is traversed one way when t is increasing and the other way when x is increasing, that may cause the negative to appear. To avoid the confusion with the sign, you can compute the integral and take the **absolute value** of your answer.

The arc length of the parametric curve $x = x(t)$ and $y = y(t)$ on interval $t \in [t_1, t_2]$ can be computed by integrating the length element ds from t_1 to t_2. The length element ds on a sufficiently small interval can be approximated by the hypotenuse of a triangle with sides dx and dy. Thus $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{(x'dt)^2 + (y'dt)^2} = \sqrt{((x')^2 + (y')^2)dt^2} = \sqrt{(x')^2 + (y')^2}\, dt$ and so

$$L = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Careful: the bounds are t-bounds, not the x-bounds.

The surface area of the surface of revolution of the parametric curve $x = x(t)$ and $y = y(t)$ for $t_1 \leq t \leq t_2$.

a) For the revolution about x-axis, integrate the surface area element dS which can be approximated as the product of the circumference $2\pi y$ of the circle with radius y and the height that is given by the arc length element ds. Since ds is $\sqrt{(x')^2 + (y')^2}dt$, the formula that computes the surface area is

$$S_x = \int_{t_1}^{t_2} 2\pi y \sqrt{(x')^2 + (y')^2}dt.$$

b) For the revolution about y-axis, the surface element dS can be approximated as the product of $2\pi x$ and the arc length element $ds = \sqrt{(x')^2 + (y')^2}dt$. Thus,

$$S_y = \int_{t_1}^{t_2} 2\pi x \sqrt{(x')^2 + (y')^2}dt.$$

Practice Problems.

1. Sketch the curve and indicate the direction in which the curve is traced as the parameter increases (you can use **TRACE** key to see that). Then eliminate the parameter to find a Cartesian equation of the curve (i.e. $y = y(x)$ format).

   (a) $x = -4t + 4, \;\; y = 2t + 5, \;\; 0 \leq t \leq 2.$ (b) $x = t^2, \;\; y = 6 - 3t.$

   (c) $x = t^2, \;\; y = 6 - 3t, \;\; 0 \leq t \leq 2.$ (d) $x = e^t, \;\; y = e^{-t}.$

   (e) $x = 2 + 2\cos t, \;\; y = 2\sin t, \;\; 0 \leq t \leq \pi.$ (f) $x = 2 + 2\cos t, \;\; y = 2\sin t, \;\; 0 \leq t \leq 3\pi.$

2. Find the first and second derivatives. Then find an equation of the tangent to the curve at the point corresponding to the given value of parameter t.

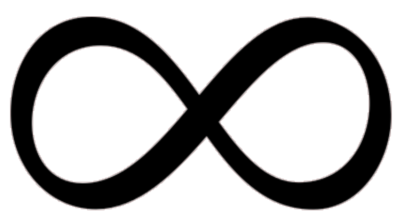
   (a) $x = t - t^3, \;\; y = 2 - 4t, \;\; t = 1.$ (b) $x = e^t, \;\; y = e^{-t}, \;\; t = 0.$

3. Find an equation of the tangent to the curve at the given point.

   (a) $x = t^2, \;\; y = 6 - 3t, \;\; (9, 15).$ (b) $x = 2 + 2\cos t, \;\; y = 2\sin t, \;\; (2, -2).$

4. Find the points on the given curve where tangent is horizontal and vertical.

   (a) $x = t^2, \;\; y = t^3 - 3t.$ (b) $x = \cos t, \;\; y = \cos t \sin t.$



5. Find the point of self-intersection and find the equations of the two tangents at that point.

(a) $x = t^2, \ \ y = t^3 - 3t.$ (b) $x = \cos t, \ \ y = \cos t \sin t.$

6. Find the area bounded by the given curve(s).

(a) $x = t^2, \ \ y = t^3 - 3t.$
(b) $x = \cos t, \ \ y = \cos t \sin t.$
(c) $x = \sin t, \ \ y = \cos^2 t \sin t$ and x-axis.
(d) $x = t - \frac{1}{t}, \ \ y = t + \frac{1}{t}$, and $y = 2.5$.

7. Find the length of the curve.

(a) $x = t^3, \ \ y = t^2$ for $0 \leq t \leq 4$.
(b) $x = 2 + 2\cos t, \ \ y = 2\sin t$ from (4, 0) to (0, 0).
(c) $x = 1 + e^{-t}, \ \ y = t^2, \ \ -2 \leq t \leq 2$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two digits.
(d) $x = \ln t, \ \ y = e^{-t}, \ \ 1 \leq t \leq 2$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two digits.

8. Find the length of the loop of the given curve. Use the Left-Right Sums program with 100 steps to approximate the integral.

(a) $x = t^2, \ \ y = t^3 - 3t.$ (b) $x = 3t - t^3, \ \ y = t^2.$

9. Find the area of the surface obtained by rotating the given curve about the specified line.

(a) $x = t^3, \ \ y = t^2, \ \ 0 \leq t \leq 1$, about x-axis.
(b) $x = 2 + 2\cos t, \ \ y = 2\sin t$, from (4, 0) to (0,0) about x-axis.
(c) $x = 3t^2, \ \ y = 2t^3$, from (0,0) to (3,2), about y-axis.
(d) $x = t + t^3, \ \ y = t - \frac{1}{t^2}, \ \ 1 \leq t \leq 2$, about x-axis. Use the Left-Right Sums program to approximate the value of the integral computing the surface area to the first two digits.
(e) $x = t + t^3, \ \ y = t - \frac{1}{t^2}, \ \ 1 \leq t \leq 2$, about y-axis. Average the Left and the Right Sums with 100 steps to approximate the value of the integral computing the surface area.

Solutions.

1. (a) Graph $x = -4t + 4$, $y = 2t + 5$ and note that the graph is a line. When $t = 0$, $x = -4(0)+4 = 4$ and $y = 2(0)+5 = 5$. When $t = 2$, $x = -4(2)+4 = -4$ and $y = 2(2)+5 = 9$. So, this is a line segment from (4,5) to (-4, 9). The Cartesian equation of this line can be obtained by solving the first equation for t (get $t = \frac{x-4}{-4}$) and plugging that in the second equation. Obtain $y = 2\frac{x-4}{-4} + 5 = \frac{-1}{2}x + 7$. Note that when t is increasing from 0 to 2, x is decreasing from 4 to -4. So, the positive direction of t corresponds to the negative direction of x.

(b) Graph $x = t^2$, $y = 6 - 3t$ and note that this is a parabola with vertex on y-axis. The curve is traversed so that the y-values decrease when t-values increase. Solving the first equation for x gives you $t = \pm\sqrt{x}$. Plugging that in the second equation yields $y = 6 \mp 3\sqrt{x}$.

(c) This is the same curve as in part (b). When $t = 0$, $x = 0^2 = 0$ and $y = 6 - 3(0) = 6$. When $t = 2$, $x = 2^2 = 4$ and $y = 6 - 3(2) = 0$. So, this is the part of the parabola from (b) from (0,6) to (4,0).

(d) $x = e^t \Rightarrow t = \ln x$ (note that $x = e^t$ is positive for every t). Plug that in y-equation to get $y = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$. So, this is the hyperbola $y = \frac{1}{x}$ for $x > 0$. The curve is traversed so that the x-values increase and y-values decrease when t-values increase.

(e) $x = 2 + 2\cos t$ and $y = 2\sin t$ is the circle of radius 2 centered at (2,0). The Cartesian equation of this circle is $(x-2)^2 + y^2 = 4$. The curve is traversed counter-clockwise which is the positive direction. When $t = 0$, $x = 2 + 2\cos(0) = 2 + 2 = 4$ and $y = 2\sin(0) = 0$. When $t = \pi$, $x = 2 + 2\cos(\pi) = 2 - 2 = 0$ and $y = 2\sin\pi = 0$. So, it is the upper half of the circle traversed in the positive direction.

(f) This is the same circle as in part (e). When $t = 0$, $x = 4$ and $y = 0$. When $t = 3\pi$, $x = 2 + 2\cos(3\pi) = 2 - 2 = 0$ and $y = 2\sin 3\pi = 0$. This indicates one full rotation and one half of the full rotation from (4,0) to (0,0).

2. (a) $x = t - t^3, y = 2 - 4t \Rightarrow dx = (1 - 3t^2)dt$ and $dy = -4dt$. So, $\frac{dy}{dx} = \frac{-4}{1-3t^2}$. When $t = 1$, $x = 1 - 1^3 = 0$ and $y = 2 - 4(1) = -2$. So, the tangent passes (0,-2). The slope of the tangent is obtained by plugging $t = 1$ in the derivative $\frac{dy}{dx} = \frac{-4}{1-3t^2}$. So $m = \frac{-4}{1-3(1)^2} = \frac{-4}{-2} = 2$. The equation of the tangent is $y = 2x - 2$.

   The second derivative can be obtained as derivative of the first derivative $\frac{dy}{dx} = \frac{-4}{1-3t^2}$ divided by $dx = (1 - 3t^2)dt$. So, $\frac{d^2y}{dx^2} = \frac{-4(-1)(1-3t^2)^{-2}(-6t)}{1-3t^2} = \frac{-24t}{(1-3t^2)^3}$.

   (b) $x = e^t, y = e^{-t} \Rightarrow dx = e^t dt$ and $dy = -e^{-t}dt$. So, $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = \frac{-1}{e^{2t}}$. When $t = 0$, $x = e^0 = 1$ and $y = e^{-0} = 1$. So, the tangent passes (1,1). The slope of the tangent is obtained by plugging $t = 0$ in the derivative $\frac{dy}{dx}$. So $m = \frac{-1}{e^0} = -1$. The equation of the tangent is $y - 1 = -1(x - 1) \Rightarrow y = -x + 2$.

   The second derivative can be obtained as derivative of the first derivative $\frac{dy}{dx} = -e^{-2t}$ divided by $dx = e^t dt$. So, $\frac{d^2y}{dx^2} = \frac{2e^{-2t}}{e^t} = 2e^{-3t}$.

3. (a) Find the t-value that corresponds to (9,15). Note that you need that value because you will plug it in the derivative to get the slope.

   Set $(x, y) = (9, 15) \Rightarrow x = t^2 = 9$, and $y = 6 - 3t = 15$. From the first equation, $t = \pm 3$ Just one of these two values will work in the the second equation. Either solving the second equation for t ($6 - 3t = 15 \Rightarrow -3t = 9 \Rightarrow t = -3$) or plugging both ± 3 into the second equation to see which one works will produce $t = -3$.

   The first derivative is $\frac{dy}{dx} = \frac{-3dt}{2tdt} = \frac{-3}{2t}$. Thus, the slope is $m = \frac{-3}{2(-3)} = \frac{1}{2}$. The equation of the tangent is $y - 15 = \frac{1}{2}(y - 9) \Rightarrow y = \frac{1}{2}x + \frac{21}{2}$.

   (b) Find the t-value that corresponds to (2,-2). Set $(x, y) = (2, -2) \Rightarrow x = 2 + 2\cos t = 2$, and $y = 2\sin t = -2$. From the first equation, $\cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$. *Careful* not to conclude that $t = \frac{\pi}{2}$ just because it is the calculator answer for $\cos^{-1}(0)$. You have to take the second equation into consideration too. From the second equation $\sin t = -1 \Rightarrow t = \frac{-\pi}{2}$.

   The first derivative is $\frac{dy}{dx} = \frac{2\cos t dt}{-2\sin t dt} = \frac{-\cos t}{\sin t}$. Thus, the slope is $m = \frac{-\cos(\frac{-\pi}{2})}{\sin(\frac{-\pi}{2})} = \frac{0}{-1} = 0$. The equation of the tangent is $y + 2 = 0(x - 2) \Rightarrow y = -2$.

4. (a) Graph the curve first. Note that using either **zoom standard** or **zoom fit** you will not be able to see the loop of this curve. To see the loop, you can change $\mathbf{T_{min}}$ in your standard window to be a negative number (for example anything smaller than -3 will work out nicely in this case). The graph looks like a ribbon. The loop has counter-clockwise orientation. From the graph we can see that there are two horizontal and one vertical tangent.

   $x = t^2 \Rightarrow dx = 2tdt$ and $y = t^3 - 3t \Rightarrow dy = (3t^2 - 3)dt$. Thus $\frac{dy}{dx} = \frac{3t^2-3}{2t}$. The curve has horizontal tangents at points at which $\frac{dy}{dx} = 0 \Rightarrow dy = 0 \Rightarrow 3t^2 - 3 = 0 \Rightarrow 3t^2 = 3 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$. Plug the two t-values in $x = t^2$ and $y = t^3 - 3t$ to get the coordinates of two points with horizontal tangents. $t = 1 \Rightarrow x = 1$ and $y = 1 - 3 = -2$. $t = -1 \Rightarrow x = 1$ and $y = -1 + 3 = 2$. So the points are (1, 2) and (1, -2).

   The curve has a vertical tangent at a point at which $\frac{dy}{dx}$ is not defined $\Rightarrow dx = 0 \Rightarrow 2t = 0 \Rightarrow t = 0$. When $t = 0$, $x = 0^2 = 0$ and $y = 0^3 - 3(0) = 0$. So, at the point (0,0) there is a vertical tangent.

   (b) The curve looks like an infinity symbol traversed counter-clockwise. From the graph we can see that there are four points with horizontal and two with vertical tangent. $x = \cos t \Rightarrow dx = -\sin t dt$ and $y = \cos t \sin t \Rightarrow dy = (-\sin^2 t + \cos^2 t)dt$.

   For the horizontal tangents $\frac{dy}{dx} = 0 \Rightarrow dy = 0 \Rightarrow -\sin^2 t + \cos^2 t = 0 \Rightarrow \cos^2 t = \sin^2 t \Rightarrow \cos t = \pm \sin t \Rightarrow 1 = \tan t$ and $-1 = \tan t \Rightarrow t = \pm\frac{\pi}{4}$, $t = \pm\frac{3\pi}{4}$. Plugging the four t-values into the x and y equations, you obtain the coordinates of four points on the curve with horizontal tangents $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, and $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

   For the vertical tangents $\frac{dy}{dx}$ is not defined $\Rightarrow dx = 0 \Rightarrow -\sin t = 0 \Rightarrow t = 0$ and $t = \pi$. $t = 0 \Rightarrow (x, y) = (1, 0)$ and $t = \pi \Rightarrow (x, y) = (-1, 0)$.

5. (a) From the graph you can see that the self-intersection is at the point that is on the x-axis. The x-axis has the equation $y = 0$. So, to find the self-intersection, set y to 0 and solve for t. *Careful:* don't set x equal to y in order to find the self-intersection.

   $y = t^3 - 3t = 0 \Rightarrow t(t^2 - 3) = 0 \Rightarrow t = 0$, $t^2 = 3 \Rightarrow t = 0$, $t = \pm\sqrt{3}$. When $t = 0$, then $x = 0$ and $y = 0$. This point corresponds to the origin and from the graph you can see that this is not the self-intersection. So, $t = \pm\sqrt{3}$ are the values you need. You can think of $t = -\sqrt{3}$ as the time when an object positioned at (x, y) enters the loop and of $t = \sqrt{3}$ as the time when it leaves the loop. Obtain the (x, y)-coordinate by plugging t-values in x and y equations. $t = \pm\sqrt{3} \Rightarrow x = 3$ and $y = 0$. So, the point is (3,0).

   To find the two slopes of the tangents at this point, plug $t = \pm\sqrt{3}$ into the derivative $\frac{dy}{dx} = \frac{3t^2-3}{2t}$.

   $t = \sqrt{3} \Rightarrow m = \frac{9-3}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$ or 1.73. So, the tangent is $y - 0 = \sqrt{3}(x - 3) \Rightarrow y = \sqrt{3}x - 3\sqrt{3}$.

   $t = -\sqrt{3} \Rightarrow m = \frac{9-3}{-2\sqrt{3}} = \frac{-3}{\sqrt{3}} = -\sqrt{3}$ or -1.73. So, the tangent is $y - 0 = -\sqrt{3}(x - 3) \Rightarrow y = -\sqrt{3}x + 3\sqrt{3}$.

   (b) From the graph, you can see that the self-intersection is at the origin (0,0). You need to find (at least) two t-values that correspond to those points. So, solve the equations $x = 0$ and $y = 0$ for t. From the first, you have $t = \pm\frac{\pi}{2}$. Those values produce 0 when plugged in the second equation, so those are the values you can use.

   To find the two slopes of the tangents at this point, plug $t = \pm\frac{\pi}{2}$ into the derivative $\frac{dy}{dx} = \frac{-\sin^2 t + \cos^2 t}{-\sin t}$.

$t = \frac{\pi}{2} \Rightarrow m = \frac{-1}{-1} = 1$. So, the tangent is $y - 0 = 1(x-0) \Rightarrow y = x$.
$t = -\frac{\pi}{2} \Rightarrow m = \frac{-1}{1} = -1$. So, the tangent is $y - 0 = -1(x-0) \Rightarrow y = -x$.

6. (a) In the previous problem we have found that the t-values at the self-intersection are $\pm\sqrt{3}$. Those t-values bound all the t-values in the loop and give you the bounds of the integration. Thus, $A = \pm\int_{-\sqrt{3}}^{\sqrt{3}} ydx = \pm\int_{-\sqrt{3}}^{\sqrt{3}}(t^3-3t)(2t)dt = \pm\int_{-\sqrt{3}}^{\sqrt{3}}(2t^4-6t^2)dt = \pm(\frac{2}{5}t^5-2t^3)|_{-\sqrt{3}}^{\sqrt{3}} = \pm(-8.313)$. Thus, the area is $A = 8.314$.

   (b) The area can be found as the double of the area of a single loop. The bounds are again the self-intersection t-values $\pm\frac{\pi}{2}$. Thus $A = \pm 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ydx = \pm 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin t(-\sin t)dt = \mp 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin^2 t dt$. Evaluate this integral using the substitution $u = \sin t$ (this is the "good case"). Get $\mp 2\frac{\sin^3 t}{3}|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \mp 2(\frac{1}{3}+\frac{1}{3}) = \frac{4}{3}$.

   (c) The curve looks like the top part of the infinity symbol for $x > 0$ and the bottom part for $x < 0$. The area can be found again as double of the area of just one of those two parts. Note from the graph that the two points bounding the relevant part of the curve are on the x-axis so $y = 0$. Thus, you need to find (at least) two (consecutive) t-values that are solutions of $y = 0$. You can take $t = 0$ and $t = \frac{\pi}{2}$. So $A = \pm 2\int_0^{\frac{\pi}{2}} ydx = \pm 2\int_0^{\frac{\pi}{2}} \cos^2 t \sin t \cos t dt = \pm 2\int_0^{\frac{\pi}{2}} \cos^3 t \sin t dt$. Evaluate this integral using the substitution $u = \cos t$ (this is the "good case"). Get $\mp 2\frac{\cos^4 t}{4}|_0^{\frac{\pi}{2}} = \mp 2(0-\frac{1}{4}) = \frac{1}{2}$.

   (d) To find the bounds, set two different y equations equal and solve for t. $t + \frac{1}{t} = 2.5 \Rightarrow t^2 - 2.5t + 1 = 0 \Rightarrow t = 2$ and $t = \frac{1}{2}$. From the graph, you can see that $y = 2.5$ is upper curve. So $A = \pm\int_{1/2}^{2}(2.5 - t - \frac{1}{t})(1+\frac{1}{t^2})dt = \pm\int_{1/2}^{2}(2.5 - t - \frac{1}{t} + \frac{2.5}{t^2} - \frac{1}{t} - \frac{1}{t^3})dt = (2.5t - \frac{t^2}{2} - 2\ln t - \frac{2.5}{t} + \frac{1}{2t^2})|_{1/2}^{2} = .977$.

7. (a) $x = t^3 \Rightarrow x' = 3t^2$, $y = t^2 \Rightarrow y' = 2t$. The length elements is $ds = \sqrt{(3t^2)^2 + (2t)^2}dt = \sqrt{9t^4+4t^2}dt = \sqrt{9t^2+4}\ tdt$ The bounds are $0 \leq t \leq 4$ so the length is $L = \int_0^4 \sqrt{9t^2+4}\ tdt$. Using the substitution $u = 9t^2 + 4$ obtain that $L = \frac{1}{18}\frac{2}{3}(9t^2+4)^{3/2}|_0^4 = \frac{1}{27}(148^{3/2} - 8) = 66.38$.

   (b) You need to find the t-bounds. When $(x, y) = (4, 0)$, $x = 2 + 2\cos t = 4 \Rightarrow \cos t = 1 \Rightarrow t = 0$ and $y = 2\sin t = 0 \Rightarrow t = 0$ and π. The value $t = 0$ agrees with the first equation so that is the lower bound. When $(x, t) = (0, 0)$, $x = 2 + 2\cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = \pi$ and $y = 2\sin t = 0 \Rightarrow t = 0$ and π. The value $t = \pi$ agrees with the first equation so that is the upper bound.
   $x = 2 + 2\cos t = 4 \Rightarrow x' = -2\sin t$ and $y = 2\sin t \Rightarrow y' = 2\cos t$. The length is $L = \int_0^\pi \sqrt{(-2\sin t)^2 + (2\cos t)^2}dt = \int_0^\pi \sqrt{4\sin^2 t + 4\cos^2 t}dt = \int_0^\pi \sqrt{4}dt = 2\int_0^\pi dt = 2\pi$.

   (c) Write down the integral you need to evaluate *before* using the program. $x = 1 + e^{-t} \Rightarrow x' = -e^{-t}$ and $y = t^2 \Rightarrow y' = 2t$. The length is $L = \int_{-2}^{2}\sqrt{(-e^{-t})^2 + (2t)^2}dt = \int_{-2}^{2}\sqrt{e^{-2t} + 4t^2}dt$. Now that you know which integral you need, you can switch your calculator back to the functions mode and use the program with $Y_1 = \sqrt{e^{-2x} + 4x^2}$, $a = -2$, $b = 2$. With $n = 100$, you have that the Left Sum is 11.94 and the Right Sum is 11.77. Since both round to 12, conclude that the length is approximately 12.

   (d) $x = \ln t \Rightarrow x' = \frac{1}{t}$ and $y = e^{-t} \Rightarrow y' = -e^{-t}$. $L = \int_1^2 \sqrt{\frac{1}{t^2} + e^{-2t}}dt$. Using the Left-Right Sums program with $Y_1 = \sqrt{\frac{1}{x^2} + e^{-2x}}$, $a = 1$, $b = 2$, and $n = 100$ you obtain the length of

.73.

8. (a) This is the same "ribbon curve" we have worked with in previous problems. We have determined that $t = \pm\sqrt{3}$ correspond to the self-intersection t-values. Note that we used the same bounds to find the area of the loop. So, the length is $L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (3t^2-3)^2}dt$ or $\int_0^2 \sqrt{4t^2 + (3t^2-3)^2}dt$. Using the program with $Y_1 = \sqrt{4x^2 + (3x^2-3)^2}$, $a = -\sqrt{3}$, $b = \sqrt{3}$, and $n = 100$, you obtain the length of $L = 10.74$.

   (b) Graph the curve first. It is a "ribbon curve" with the self-intersection on the y-axis that has the equation $x = 0$. So, the corresponding t-values can be found by setting x equal to 0. $x = 0 \Rightarrow 3t - t^3 = 0 \Rightarrow t(3-t^2) = 0 \Rightarrow t = 0, 3 = t^2 \Rightarrow t = 0$ and $t = \pm\sqrt{3}$. When $t = 0$, $x = 0$ and $y = 0$ and this point is not the self-intersection. So $t = \pm\sqrt{3}$ are the t-values needed. So, the length is $L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(3t^2-3)^2 + (2t)^2}dt$ or $\int_0^2 \sqrt{(3t^2-3)^2 + 4t^2}dt$. Using the program with $Y_1 = \sqrt{(3x^2-3)^2 + 4x^2}$, $a = -\sqrt{3}$, $b = \sqrt{3}$, and $n = 100$, you obtain the length of $L = 10.74$.

9. (a) $x = t^3 \Rightarrow x' = 3t^2$ and $y = t^2 \Rightarrow y' = 2t$. $S_x = \int_0^1 2\pi t^2\sqrt{9t^4 + 4t^2}dt$. Simplify before integrating. $S_x = 2\pi\int_0^1 t^2\sqrt{9t^2+4}\,tdt$. Use the substitution $u = 9t^2 + 4 \Rightarrow du = 18tdt$. Note that the term t^2 substitutes as $t^2 = \frac{u-4}{9}$ that can be obtained by solving $u = 9t^2 + 4$ for t^2. Thus, the integral becomes $S_x = 2\pi\int_0^1 \frac{u-4}{9}\sqrt{u}\frac{du}{18} = \frac{\pi}{81}\int_0^1 (u-4)\sqrt{u}du = \frac{\pi}{81}\int_0^1 (u^{3/2} - 4u^{1/2})du = \frac{\pi}{81}(\frac{2}{5}(9t^2+4)^{5/2} - \frac{8}{3}(9t^2+4)^{3/2})|_0^1 = \frac{\pi}{81}(\frac{2}{5}13^{5/2} - \frac{8}{3}13^{3/2} - \frac{64}{5} + \frac{64}{3}) = 4.936$.

   (b) $x = 2 + 2\cos t \Rightarrow x' = -2\sin t$ and $y = 2\sin t \Rightarrow y' = 2\cos t$. The length element is $ds = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$. You need to find the t-bounds that correspond to (4, 0) to (0,0). Similarly as in problem 7 (b) obtain that the corresponding t-values are 0 and π. So $S_x = \int_0^\pi 2\pi 2\sin t\, 2dt = 8\pi(-\cos t)|_0^\pi = 16\pi$.

   (c) Compute the length element to be $ds = \sqrt{36t^2 + 36t^4}dt = \sqrt{1+t^2}\,6tdt$ and the t-values corresponding to (0,0) and (3,2) to be $t = 0$ and $t = 1$. So, $S_y = \int_0^1 2\pi 3t^2\sqrt{1+t^2}\,6tdt = 36\pi\int_0^1 t^2\sqrt{1+t^2}\,tdt$. Using the substitution $u = 1+t^2$ obtain that $du = 2tdt$ and $t^2 = u-1$. So, $S_y = 18\pi\int_0^1 (u-1)\sqrt{u}du = 18\pi(\frac{2}{5}(1+t^2)^{5/2} - \frac{2}{3}(1+t^2)^{3/2})|_0^1 = 18\pi(\frac{2}{5}2^{5/2} - \frac{2}{3}2^{3/2} - \frac{2}{5} + \frac{2}{3}) = 36.405$.

   (d) $S_x = \int_1^2 2\pi(t - \frac{1}{t^2})\sqrt{(1+3t^2)^2 + (1+2t^{-3})^2}dt$. Careful when typing this formula in the calculator. Using the program with $n = 200$ subintervals obtain that the Left Sum is 58.74 and the Right Sum is 59.46. Thus the surface area is approximately $S_x = 59$.

   (e) $S_y = \int_1^2 2\pi(t + t^3)\sqrt{(1+3t^2)^2 + (1+2t^{-3})^2}dt$. Using the program with $n = 100$ obtain that the Left Sum is 303.71 and the Right Sum is 311.29. They average to the surface area of $S_y = 307.5$.