Calculus 2
Lia Vas

Partial Fractions

A rational function is a quotient of two polynomial functions. The method of partial fractions is a general method for evaluating integrals of rational function. The idea behind this method is to write a rational function as a sum of simpler rational functions called **the partial fractions** and then to integrate each term.

Case 1. Let us consider first rational functions of the form $\frac{p(x)}{q(x)}$ where p and q are polynomials such that **the degree of p is smaller than the degree of q**. In this case, to find the partial fractions:

1. Factor the denominator into a product of powers of linear terms $ax+b$ and quadratic terms ax^2+bx+c. The quadratic equation $ax^2+bx+c=0$ should have no real solutions otherwise you would be able to factor ax^2+bx+c into a product of two linear terms.

2. For each power of a linear term of the form $(ax+b)^k$, introduce k partial fractions with unknown coefficients $A_1, A_2, \ldots A_k$:
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k}$$

3. For each power of a quadratic term of the form $(ax^2+bx+c)^k$, introduce k partial fractions with unknown coefficients $A_1, A_2, \ldots A_k$ and $B_1, B_2, \ldots B_k$:
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

4. Determine the unknown coefficients by combining the partial fractions into a single fraction. Be careful when finding the least common denominator: note that it should be *equal to the initial denominator* $q(x)$. Then equate the coefficients of the numerator you obtain with the coefficients of the initial numerator $p(x)$. This should give you a system in all the unknown coefficients. Note that *the number of equations should match the number of unknowns.*

   When you determine the unknown coefficients, you have found the partial fractions you need for the integration.

5. Write the given rational function as a sum of partial fractions from the previous step and integrate each partial fraction.

Case 2. If the rational function is of the form $\frac{p(x)}{q(x)}$ where p and q are polynomials such that **the degree of p is greater or equal to the degree of q**, then use the long division to divide the polynomial p by the polynomial q. If s is the resulting quotient and r is the remainder, then you can write
$$\frac{p}{q} = s + \frac{r}{q}.$$

The polynomial s can be easily integrated term-by-term. The degree of r will be smaller than the degree of q so the rational function $\frac{r}{q}$ falls in case 1 category. Thus, you can reduce case 2 into case 1.

Practice Problems.

a) Write out the form of the partial fractions of the given function (do not determine the constants).

1. $\dfrac{2x-3}{x^2-1}$ 2. $\dfrac{2x-3}{(x^2-1)(x+2)}$ 3. $\dfrac{2x-3}{(x-1)^2(x+1)}$

4. $\dfrac{2x-3}{(x-1)^3(x+1)^2}$ 5. $\dfrac{2x-3}{(x-1)(x^2+1)}$ 6. $\dfrac{2x-3}{(x-1)^2(x^2+1)^2}$

b) Evaluate the integrals.

1. $\displaystyle\int \frac{2x-3}{x^2-1}\,dx$ 2. $\displaystyle\int \frac{x-9}{x^2+3x-10}\,dx$ 3. $\displaystyle\int \frac{x^2+1}{x^2-x}\,dx$ 4. $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

5. $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$ 6. $\displaystyle\int \frac{x^3}{x^2+1}\,dx$ 7. $\displaystyle\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx$

Solutions. a)

1. The denominator factors as $(x-1)(x+1)$. Since you have two linear terms (both on power 1), you need just two partial fractions, one for each term. So, the function decomposes as $\frac{A}{x-1}+\frac{B}{x+1}$.

2. The denominator factors as $(x-1)(x+1)(x+2)$. So, you have three linear terms (all on power 1) and you need three partial fractions. The function decomposes as $\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+2}$.

3. The denominator factors as $(x-1)^2(x+1)$. So you have two linear terms, the first one is with power 2 and the second with power 1. So, for the first linear term, you need two fractions $\frac{A}{x-1}$ and $\frac{B}{(x-1)^2}$). For the second linear term, you need another fraction $\frac{C}{x+1}$. So, the function decomposes as $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$.

4. There are two linear terms, the first one is with power 3 and the second with power 2. So, the function decomposes as $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+1}+\frac{E}{(x+1)^2}$.

5. There is a linear term and a quadratic term (note that x^2+1 cannot be factored further since $x^2+1=0$ has no real solutions). You need a linear term in the denominator of the fraction with x^2+1 in denominator. So, the function decomposes as $\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$.

6. The function decomposes as $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{(x^2+1)^2}$.

b)

1. By problem 1. in part a), the function has the partial fractions decomposition as $\frac{A}{x-1} + \frac{B}{x+1}$. The common denominator is the product $(x-1)(x+1)$ so that $\frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} = \frac{Ax+A+Bx-B}{(x-1)(x+1)}$. Note that this fraction should be equal to the initial function (note that the denominators are equal). Thus

$$\frac{Ax + A + Bx - B}{(x-1)(x+1)} = \frac{2x-3}{(x-1)(x+1)} \quad \Rightarrow \quad Ax + A + Bx - B = 2x - 3 \quad \Rightarrow$$

linear terms (with x) are equal $\Rightarrow A + B = 2$
free terms (with no x) are equal $\Rightarrow A - B = -3$.

Note that now we obtained a system of two equations with two unknowns (note that number of equations matches the number of unknowns). Using eliminations of variables, you can solve for one variable in one equation and substitute that in the other equation. For example $B = 2 - A$ using the first equation. The second equation becomes $A - 2 + A = -3 \Rightarrow 2A = -1 \Rightarrow A = \frac{-1}{2}$. Then $B = 2 - A = 2 + \frac{1}{2} = \frac{5}{2}$.

Finally, using partial fractions the integral becomes $\int \frac{2x-3}{(x-1)(x+1)}\, dx = \int \frac{-1/2}{x-1} dx + \int \frac{5/2}{x+1} dx = \frac{-1}{2} \int \frac{1}{x-1} dx + \frac{5}{2} \int \frac{1}{x+1} dx = \frac{-1}{2} \ln|x-1| \frac{5}{2} \ln|x+1| + c$.

2. The denominator factors as $(x+5)(x-2)$. The partial fractions decomposition is $\frac{A}{x+5} + \frac{B}{x-2}$. Find A and B similarly as in previous problem and obtain that $A = 2$ and $B = -1$. So, the integral becomes $\int \frac{2}{x+5} dx + \int \frac{-1}{x-2} dx = 2 \ln|x+5| - \ln|x-2| + c$.

3. Note that the degree of the numerator is equal (not smaller!) than the degree of the denominator. That means that you will have to divide the polynomials first.

$$\begin{array}{r|l} & 1 \\ x^2 - x & \overline{x^2 + 1} \\ & -(x^2 - x) \\ \hline & x + 1 \end{array}$$

Thus, the quotient is 1 and the remainder is $x+1$. The function $\frac{x^2+1}{x^2-x}$ is equal to $1 + \frac{x+1}{x^2-x}$. The denominator factors as $x^2 - x = x(x-1)$ and so the fraction $\frac{x+1}{x^2-x} = \frac{x+1}{x(x-1)}$. It has the partial decomposition $\frac{A}{x} + \frac{B}{x-1}$. Find A and B similarly as in previous problem and obtain that $A = -1$ and $B = 2$. So, the integral becomes $\int \frac{x^2+1}{x^2-x} dx = \int (1 + \frac{x+1}{x^2-x}) dx = \int (1 - \frac{1}{x} + \frac{2}{x-1}) dx = x - \ln|x| + 2 \ln|x-1| + c$.

4. The denominator factors as $x^3 + 2x^2 = x^2(x+2)$. So, the partial fractions needed are $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Note that *least* common denominator is $x^2(x+2)$, and *not* the product of all three denominators $x^3(x+2)$. Thus,

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} = \frac{Ax(x+2) + B(x+2) + Cx^2}{x^2(x+2)} = \frac{Ax^2 + 2Ax + Bx + 2B + Cx^2}{x^2(x+2)}.$$

Equate this fraction with the initial function $\frac{5x^2+3x-2}{x^3+2x^2}$.

Equating the terms with x^2, you get that $A + C = 5$. So $C = 5 - A$.

Equating the terms with x, you get that $2A + B = 3$.

Equating the terms with no x, you get $2B = -2 \Rightarrow B = -1$. Plugging that in the second equation gives you $2A - 1 = 3 \Rightarrow 2A = 4 \Rightarrow A = 2$. Then obtain C from the first equation $C = 5 - A = 5 - 2 = 3$.

The integral becomes $\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \frac{2}{x}dx + \int \frac{-1}{x^2}dx + \int \frac{3}{x+2}dx = 2\ln|x| + \frac{1}{x} + 3\ln|2+x| + c$.

5. The denominator is a linear term on the third power. So, the partial fraction decomposition is

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} = \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} = \frac{Ax^2 + 2Ax + A + Bx + B + C}{(x+1)^3}.$$

Equate this fraction with the initial function $\frac{x^2}{(x+1)^3} = \frac{1x^2+0x+0}{(x+1)^3}$.

Equating the terms with x^2, you get that $A = 1$.

Equating the terms with x, you get that $2A + B = 0$. Since $A = 1$, $B = -2A = -2$.

Equating the terms with no x, you get $A + B + C = 0 \Rightarrow C = -B - A = 2 - 1 = 1$.

So, the integral is $\int \frac{x^2}{(x+1)^3}dx = \int \frac{1}{x+1}dx + \int \frac{-2}{(x+1)^2}dx + \int \frac{1}{(x+1)^3}dx = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + c$.

6. Since the degree of the numerator is larger than the degree of the denominator, you need to divide the polynomials first.

$$\begin{array}{r|l} & x \\ x^2+1 & x^3 \\ & -(x^3+x) \\ \hline & -x \end{array}$$

Thus, the quotient is x and the remainder is $-x$. The function $\frac{x^3}{x^2+1}$ is equal to $x - \frac{x}{x^2+1}$. Note that the last fraction is already in the form of a partial fraction. The integral of this fraction can be evaluated using substitution $u = x^2 + 1$. This gives you $\int \frac{x}{x^2+1}dx = \int \frac{x}{u}\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1)$. So, the initial integral is $\int \frac{x^3}{x^2+1}dx = \int (x - \frac{x}{x^2+1})dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + c$.

7. Note that x^2+1 is a quadratic term that cannot be factored in two linear terms (since $x^2+1 = 0$ has no real solutions). So, the partial fractions decomposition is $\frac{A}{x-1} + \frac{Bx+C}{x^2+1}$.

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)} = \frac{Ax^2 + A + Bx^2 - Bx + Cx - C}{(x-1)(x^2+1)}.$$

Equate this with $\frac{3x^2-4x+5}{(x-1)(x^2+1)}$.

Equating the terms with x^2, you get that $A + B = 3$. So $B = 3 - A$.

Equating the terms with x, you get that $-B + C = -4$. So $C = -4 + B = -1 - A$

Equating the terms with no x, you get $A - C = 5$. Hence $A + 1 + A = 5 \Rightarrow 2A = 4 \Rightarrow A = 2$. Thus $B = 3 - 2 = 1$ and $C = -1 - 2 = -3$.

The integral becomes $\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}dx = \int \frac{2}{x-1}dx + \int \frac{x-3}{x^2+1}dx = 2\int \frac{1}{x-1}dx + \int \frac{x}{x^2+1}dx - 3\int \frac{1}{x^2+1}dx$. The first integral is $2\ln|x-1|$. For the second, use substitution $u = x^2 + 1$ to get $\frac{1}{2}\ln(x^2+1)$. The third integral comes straight from the formula for $\tan^{-1} x$. Thus, the given integral is $2\ln|x-1| + \frac{1}{2}\ln(x^2+1) - 3\tan^{-1} x + c$.