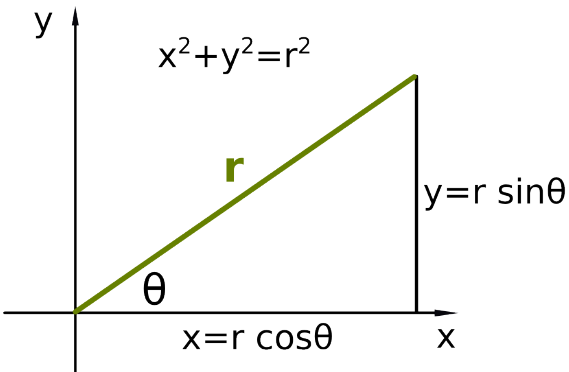
**Calculus 2**
**Lia Vas**

# Polar Coordinates

If $P=(x,y)$ is a point in the $xy$-plane and $O$ denotes the origin, let

- $r$ denote the distance from the origin $O$ to the point $P=(x,y)$. Thus, $x^2+y^2=r^2$;
- $\theta$ be the angle between the vector $\overrightarrow{OP}$ and the positive part of $x$-axis. Thus, $\tan\theta=\frac{y}{x}$.




This gives a new way to represent a point $(x,y)$. If the point is represented by $(\theta,r)$ instead of $(x,y)$, we say it is given in **polar coordinates**.

Note that $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$ so that

$$\boxed{x=r\cos\theta, \quad \text{and} \quad y=r\sin\theta}$$

In some cases, polar coordinates are more convenient to represent position than the Cartesian coordinates. Consider the example on figure below: Manhattan is build "rectangularly". Midtown Manhattan addresses given by reference to the nearest street ($x$-coordinate) and avenue $y$-coordinate are perfect for orientation. On the other hand, people populate the Burning Man camp filling the circular arches (so one's $\theta$-position would be given by an angle) having different distances from the center ($r$-coordinates).


Cartesian

Polar

A curve in Cartesian coordinate is given when one variable is a function of the other (e.g. $y=y(x)$). Analogously a curve in polar coordinates can be given as $r=r(\theta)$. Note that in this case $x$ and $y$ have **parametric equations**

$$x=r(\theta)\cos\theta \qquad\qquad y=r(\theta)\sin\theta.$$

**Example.** One of the most important examples of the polar curve is the circle. Consider the circle $x^2+y^2=a^2$. Since in polar coordinates $x^2+y^2=r^2$, this gives that $r^2=a^2 \Rightarrow r=a$ (assuming that $a$ is positive) represents the equation of this circle in the polar coordinates.

The simplicity of the equation $r=a$ illustrates the importance of polar coordinates. The equation $r=a$ represents all the points that are at distance $a$ from the origin and this matches the intuitive concept of the circle. The $(x,y)$ parametric equations

$$x=a\cos\theta \qquad\qquad y=a\sin\theta.$$

match the parametric equations of the circle discussed in the previous section.

To graph a curve in polar coordinates on your calculator, go to **Mode** and switch from **Func** to **Pol**. This will switch your calculator to the polar mode. In this mode, you can enter $r$ as a function of $\theta$ when pressing **Y=** key. Use key **X,T,$\theta$,n** to display the variable $\theta$. The standard window on your calculator is set to be $0\le\theta\le 2\pi$. In most cases, this will be adequate.

**The derivative of a polar curve.** Recall that the polar curve $r=r(\theta)$ has parametric equations $x=r(\theta)\cos\theta$ and $y=r(\theta)\sin\theta$. So, the derivative $\frac{dy}{dx}$ can be found as

$$\frac{dy}{dx}=\frac{y'(\theta)}{x'(\theta)}.$$

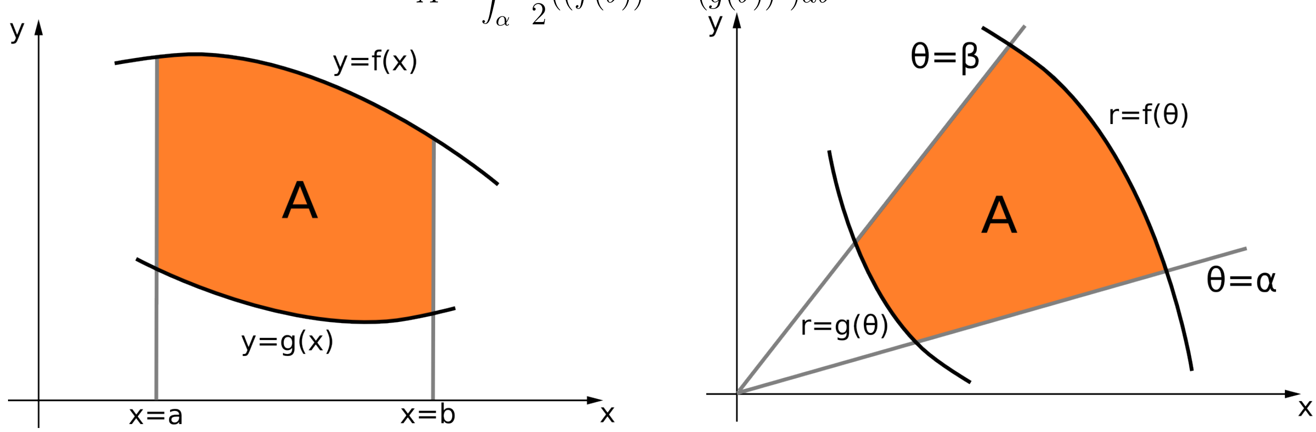
**The area bounded by the polar curve** $r=r(\theta)$ on interval $\theta\in[\alpha,\beta]$ is

$$A=\int_\alpha^\beta \frac{1}{2}(r(\theta))^2 d\theta$$

The validity of this formula will be demonstrated in Calculus 3 course.

To find the area between two polar curves $r=f(\theta)$ and $r=g(\theta)$ on interval $\theta\in[\alpha,\beta]$ determine which curve is at a larger distance from the origin and which is closer to the origin. Say that $0\le g(\theta)\le f(\theta)$ In this case, the area can be found as the difference of the two integrals $A=\int_\alpha^\beta \frac{1}{2}(f(\theta))^2 d\theta-\int_\alpha^\beta\frac{1}{2}(g(\theta))^2 d\theta$ so that

$$A=\int_\alpha^\beta \frac{1}{2}((f(\theta))^2-(g(\theta))^2)d\theta$$




Area between two curves in Cartesian and polar coordinates

**The arc length of the polar curve** $r=r(\theta)$ on interval $\theta\in[\alpha,\beta]$ can be can be computed by integrating the length element $ds$ from $\alpha$ to $\beta$. The length element $ds$ is $\sqrt{(x')^2+(y')^2}d\theta$. Substituting the derivatives of the parametric equations $x = r(\theta)\cos\theta$ and $y = r(\theta)\sin\theta$ into the formula for $ds$ and simplifying, we arrive to the formula $ds = \sqrt{((r)^2 + (r')^2)}d\theta$. So, the formula for the length is

$$L = \int_\alpha^\beta \sqrt{(r(\theta))^2 + (r'(\theta))^2}d\theta.$$

**Practice Problems.**

1. Find polar coordinates for the following set of points in Cartesian coordinates: $(2, 0), (0, 3), (-2, 0), (1, 1), (1, -1), (-1, -1)$.
2. Find Cartesian coordinates for the following set of points in polar coordinates: $(\frac{\pi}{2}, 4), (0, 5), (\pi, 4), (\frac{\pi}{4}, 2\sqrt{2}), (\frac{-\pi}{4}, 2\sqrt{2})$.
3. Sketch the following regions: (a) $r < 1$; (b) $r < 1,\ 0 \leq \theta \leq \frac{\pi}{2}$ (c) $1 < r < 3,\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.
4. A **cardioid** can be given by $r = 1 + \sin\theta$. The curves of the form $r = \sin n\theta$ and $r = \cos n\theta$ for $n = 2, 3, \ldots$ are known as **roses**. Graph the given cardioid and the roses for different values of $n$ on the calculator. Using the graphs, determine how the number of petals depends on $n$.
5. Find an equation in polar coordinates for the following set of curves in Cartesian coordinates: (a) $y = x$; (b) $x^2 + y^2 = 4$; (c) $x^2 = 4y$.
6. Find an equation in Cartesian coordinates for the following set of curves in polar coordinates: (a) $r = 3$; (b) $r = \sin\theta$; (c) $r = 4\cos\theta$.
7. Find the slope of the tangent line to the given polar curve at the point specified by the value of $\theta$. (a) $r = \frac{1}{\theta},\ \theta = \pi$; (b) $r = \cos 2\theta,\ \theta = \frac{\pi}{2}$; (c) $r = 1 + \cos\theta,\ \theta = \frac{\pi}{3}$.
8. Find the area of the region that is bounded by the given curve(s) and lies in the specified sector.
   (a) Area inside the curve $r = 4\cos\theta$.
   (b) Area inside the four-leaved rose $r = \cos 2\theta$.
   (c) Area inside the curve $r = 4\cos\theta$ and outside the curve $r = 2$.
   (d) Area inside the curve $r = 2$ and outside the curve $r = 4\cos\theta$.
   (e) Area inside the curves $r = 2$ and $r = 4\cos\theta$.
   (f) Area inside the curves $r = \sin\theta$ and $r = 2\sin\theta\cos\theta$.
   (g) Area inside the curves $r = 4\sin\theta$ and $r = 4\cos\theta$.
   (h) Area inside the curve $r = 2$ and outside the curve $r = 2\sin\theta$.
   (i) Area inside the curve $r = 4\sin(2\theta)$ and outside the curve $r = 2$.
9. Find the length of the following polar curves.
   (a) $r = 2\cos\theta,\ 0 \leq \theta \leq \frac{\pi}{2}$. (b) $r = e^{2\theta},\ 0 \leq \theta \leq \frac{\pi}{2}$. (c) $r = \theta^2,\ 0 \leq \theta \leq 2\pi$.
   (d) Find the length of the four-leaved rose $r = \cos 2\theta$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two nonzero digits.

(e) Find the length of the three-leaved rose $r = \sin 3\theta$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two nonzero digits.

**Solutions.**

1. You can determine the polar coordinates of most of these points simply by looking at the graph. $(2,0)$ is on the $x$-axis. Thus $\theta = 0$ It is at distance 2 from the origin so $r = 2$. So, $(\theta, r) = (0, 2)$. Similarly, $(x, y) = (0, 3) \Rightarrow (\theta, r) = (\frac{\pi}{2}, 3)$. $(x, y) = (-2, 0) \Rightarrow (\theta, r) = (\pi, 2)$.

   If $(x, y) = (1, 1) \Rightarrow r^2 = 1^2 + 1^1 \Rightarrow r = \sqrt{2}$. From the graph it is easy to see that $\theta = \frac{\pi}{4}$. Alternatively, find $\theta$ as $\tan^{-1}\frac{1}{1} = \frac{\pi}{4}$. So $(\theta, r) = (\frac{\pi}{4}, \sqrt{2})$. Similarly, $(x, y) = (1, -1) \Rightarrow (\theta, r) = (\frac{-\pi}{4}, \sqrt{2})$. $(x, y) = (-1, -1) \Rightarrow (\theta, r) = (\frac{5\pi}{4}, \sqrt{2})$.

2. In this problem also you can use the graph. Alternatively, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

   $(\theta, r) = (\frac{\pi}{2}, 4) \Rightarrow (x, y) = (0, 4)$, $(\theta, r) = (0, 5) \Rightarrow (x, y) = (5, 0)$, $(\theta, r) = (\pi, 4) \Rightarrow (x, y) = (-4, 0)$, $(\theta, r) = (\frac{\pi}{4} 2\sqrt{2}) \Rightarrow (x, y) = (2, 2)$, $(\theta, r) = (\frac{-\pi}{4}, 2\sqrt{2}) \Rightarrow (x, y) = (2, -2)$.

3. (a) Recall $r = 1$ represents the unit circle centered at the origin i.e. all the points that are at distance 1 from the origin. Thus, $r < 1$ represents all the points that are at a distance smaller than 1 from the origin i.e. the inside of the unit circle.

   (b) By the previous problem $r < 1$ is the inside of the unit circle. Since $0 \le \theta \le \frac{\pi}{2}$ denotes the first quadrant, the region is upper right quarter of the inside of the unit circle.

   (c) $1 < r < 3$ represents the region between the circle of radius 1 and the circle of radius 3 centered at the origin. Since $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ represents the region in the second and the third quadrant, the region is the left half of the annulus with inner radius 1 and outer radius 3.

4. Graphing the cardioid is straightforward. Let us look at the rose $r = \sin n\theta$ first. Graphing the curve for first few values of $n$, you observe the following

   | $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
   |---|---|---|---|---|---|---|
   | no. of petals | 4 | 3 | 8 | 5 | 12 | 7 |

   This indicated that this rose has $2n$ petals if $n$ is even and $n$ petals if $n$ is odd. The number of petals of the rose $y = \cos n\theta$ follows the same pattern.

5. (a) $y = x \Rightarrow r\sin\theta = r\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$. The expression $\theta = \frac{\pi}{4}$ represents the half line of $y = x$ in the first and the expression $\theta = \frac{5\pi}{4}$ represents the half line of $y = x$ in the third quadrant.

   (b) $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$.

   (c) $x^2 = 4y \Rightarrow r^2\cos^2\theta = 4r\sin\theta \Rightarrow r\cos^2\theta = 4\sin\theta \Rightarrow r = \frac{4\sin\theta}{\cos^2\theta}$ is the equation of this parabola in polar coordinates.

6. (a) $r = 3$ is the equation of the circle centered at the origin of radius 3. In Cartesian coordinates is it $x^2 + y^2 = 9$.

   (b) Multiply the equation $r = \sin\theta$ by $r$ to get $r^2 = r\sin\theta$ so that the left side becomes $x^2 + y^2$ and the right side becomes $y$. Thus, the equation is $x^2 + y^2 = y$. Note that this is the circle centered at $(0, \frac{1}{2})$ of radius $\frac{1}{2}$.

   (c) $r = 4\cos\theta \Rightarrow r^2 = 4r\cos\theta \Rightarrow x^2 + y^2 = 4x$ or $y = \pm\sqrt{4x - x^2}$. Note that this is the circle centered at (2,0) of radius 2.

7. (a) $r=\frac{1}{\theta}\Rightarrow x=\frac{1}{\theta}\cos\theta=\frac{\cos\theta}{\theta}$, $y=\frac{1}{\theta}\sin\theta=\frac{\sin\theta}{\theta}$. So $\frac{dy}{dx}=\frac{\frac{\theta\cos\theta-\sin\theta}{\theta^2}}{\frac{-\theta\sin\theta-\cos\theta}{\theta^2}}=\frac{\theta\cos\theta-\sin\theta}{-\theta\sin\theta-\cos\theta}$. At $\theta=\pi$, $\frac{dy}{dx}=-\pi$.

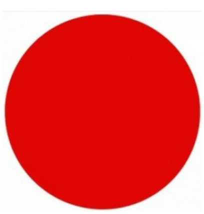
   (b) $r=\cos 2\theta\Rightarrow x=\cos 2\theta\cos\theta$, $y=\cos 2\theta\sin\theta$. So $\frac{dy}{dx}=\frac{-2\sin 2\theta\sin\theta+\cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}$. At $\theta=\frac{\pi}{2}$, $\frac{dy}{dx}=\frac{0}{1}=0$.

   (c) $r=1+\cos\theta,\Rightarrow x=(1+\cos\theta)\cos\theta$, $y=(1+\cos\theta)\sin\theta$. So $\frac{dy}{dx}=\frac{-\sin\theta\sin\theta+(1+\cos\theta)\cos\theta}{-\sin\theta\cos\theta-(1+\cos\theta)\sin\theta}$. At $\theta=\frac{\pi}{3}$, $\frac{dy}{dx}=\frac{-\frac{3}{4}+\frac{3}{4}}{-\frac{\sqrt{3}}{4}-\frac{3\sqrt{3}}{4}}=0$.

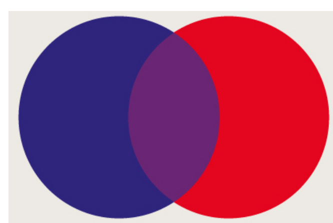
8. (a) Graph the curve first. This is the circle centered on the $x$-axis. Note that $y$-axis is tangent to the circle. The lower half of $y$-axis has the equation $\theta=\frac{-\pi}{2}$ and the upper part has the equation $\theta=\frac{\pi}{2}$. Alternatively, note that the tangent is when $r=0$. Solving $r=4\cos\theta=0$ for $\theta$ you get that $\theta=\pm\frac{\pi}{2}$. So, the area can be found as $A=\int_{-\pi/2}^{\pi/2}\frac{1}{2}(4\cos\theta)^2d\theta=8\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta$. This is the "bad case". Using the identity $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$ you obtain $A=4(\theta+\frac{1}{2}\sin 2\theta)|_{-\pi/2}^{\pi/2}=4\pi$.

   (b) Note that the total area is 4 times the area inside one petal. Let us look at the first petal. Similarly to the previous problem, note that this petal is bounded by two tangents to the rose and that the $\theta$-values corresponding to the limiting $r$-value $r=0$. So, the bounds can be found by solving $r=\cos 2\theta=0$ for $\theta$. Solve for $2\theta$ first. Obtain $2\theta=\pm\frac{\pi}{2}$ so $\theta=\pm\frac{\pi}{4}$. Hence $A=4\int_{-\pi/4}^{\pi/4}\frac{1}{2}\cos^2 2\theta d\theta=(\theta+\frac{1}{4}\sin 4\theta)|_{-\pi/4}^{\pi/4}=\frac{\pi}{2}$

   (c) The curves are two intersecting circles. $r=2$ is centered at the origin and $r=4\cos\theta$ on the $x$-axis. The region in question is the red crescent moon on the figure below. The bounds correspond to the smallest $\theta$-value corresponding to the intersection below the $x$-axis and the largest $\theta$-value corresponding to the intersection above the $x$-axis. To find the intersections, set the curve equal to each other and solve for $\theta$. $4\cos\theta=2\Rightarrow\cos\theta=\frac{1}{2}\Rightarrow\theta=\pm\frac{\pi}{3}$.



8 (a) circle



8 (c) "mastercard"

   Note that the curve $r=4\cos\theta$ is the outer radius and $r=2$ is the inner radius. So, the area is $A=\int_{-\pi/3}^{\pi/3}\frac{1}{2}((4\cos\theta)^2-2^2)d\theta=\int_{-\pi/3}^{\pi/3}(8\cos^2\theta-2)d\theta=(2\theta+2\sin 2\theta)|_{-\pi/3}^{\pi/3}=2\sqrt{3}+\frac{4\pi}{3}=7.65$.

   (d) The region here is the opposite crescent moon than the one in the previous problem. From the previous problem we have found that the circles intersect at $\theta=\pm\frac{\pi}{3}$. Note that from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, there is an inner and outer radius. To see that, look at a ray from the origin between $\frac{\pi}{3}$ to $\frac{\pi}{2}$ – note that it intersect both curves. Also note that and $r=2$ is outer and $r=4\cos\theta$ is inner. Let us refer to that portion as $A_1$. Thus $A_1=\int_{\pi/3}^{\pi/2}\frac{1}{2}(2^2-(4\cos\theta)^2)d\theta=\int_{\pi/3}^{\pi/2}(2-8\cos^2\theta)d\theta=(2\theta-4\theta-2\sin 2\theta)|_{\pi/3}^{\pi/2}=\frac{-\pi}{3}+\sqrt{3}=0.685$.

   From $\frac{\pi}{2}$ to $\pi$, on the other hand, a ray from the origin intersects just the curve $r=2$. So, this portion, let us call it $A_2$ is $A_2=\int_{\pi/2}^{\pi}\frac{1}{2}2^2=2\frac{\pi}{2}=\pi$.

   $A_1+A_2$ covers just the area above $x$-axis. So, the total area $A=2(A_1+A_2)=7.65$.

(e) Let us look at the part of the intersection of the two circles above $x$-axis. From previous two problems, we know that the intersection of two circles is at $\frac{\pi}{3}$. Notice how a ray from the origin between 0 and $\frac{\pi}{3}$ intersects just $r=2$ after it passes through the relevant region. And a ray from the origin between $\frac{\pi}{3}$ and $\frac{\pi}{2}$ intersects just $r=4\cos\theta$ after it passes through the relevant region. This indicates that you need two integrals, say $A_1$ and $A_2$ to find the area of this top part. Then the total area $A$ can be computed as $A=2(A_1+A_2)=2\int_0^{\pi/3}\frac{1}{2}2^2d\theta+2\int_{\pi/3}^{\pi/2}\frac{1}{2}(4\cos\theta)^2d\theta=\int_0^{\pi/3}4d\theta+16\int_{\pi/3}^{\pi/2}\cos^2\theta d\theta=\frac{4\pi}{3}+8(\theta+\frac{1}{2}\sin 2\theta)|_{\pi/3}^{\pi/2}=\frac{8\pi}{3}-2\sqrt{3}=4.91$.

(f) Graph the curves first. Note that they intersect in the first and the second quadrant. You can find the total area as two times the area in the first quadrant. Find the angle of intersection. $\sin\theta=2\sin\theta\cos\theta\Rightarrow 1=2\cos\theta\Rightarrow\cos\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$. Notice how a ray from the origin between 0 and $\frac{\pi}{3}$ intersects just $r=\sin\theta$ after it passes through the relevant region. And a ray from the origin between $\frac{\pi}{3}$ and $\frac{\pi}{2}$ intersects just $r=2\sin\theta\cos\theta$ after it passes through the relevant region. This indicates that you need two integrals, say $A_1$ and $A_2$ to find the area of this top part. Then the total area $A$ can be computed as $A=2(A_1+A_2)=\int_0^{\pi/3}\sin^2\theta d\theta+\int_{\pi/3}^{\pi/2}4\sin^2\theta\cos^2\theta d\theta=.307+.153=.46$.

(g) The intersection is $\frac{\pi}{4}$. Similarly to problem (f), you can find the area as the sum of two areas $A=A_1+A_2=\int_0^{\pi/4}\frac{1}{2}(4\sin\theta)^2d\theta+\int_{\pi/4}^{\pi/2}\frac{1}{2}(4\cos\theta)^2d\theta=1.14+1.14=2.283$.

(h) Area can be found as two times the area right from the $y$-axis. In the first quadrant, the curve $r=2$ is the outer radius and $r=2\sin\theta$ is the inner radius. The bounds are 0 to $\frac{\pi}{2}$. In the fourth quadrant, just $r=2$ is relevant and the bounds are $\frac{-\pi}{2}$ to 0. So $A=2(A_1+A_2)=2\int_{\frac{-\pi}{2}}^{0}\frac{1}{2}2^2d\theta+2\int_0^{\frac{\pi}{2}}\frac{1}{2}(2\sin\theta)^2d\theta=2\pi+\pi=3\pi$.

(i) You can compute the area as 4 times the area in the first quadrant. Note from the graph that the bounds are the intersections: $4\sin(2\theta)=2\Rightarrow\sin(2\theta)=\frac{1}{2}\Rightarrow 2\theta=\sin^{-1}(\frac{1}{2})=\frac{\pi}{6}$ and $2\theta=\pi-\sin^{-1}(\frac{1}{2})=\pi-\frac{\pi}{6}=\frac{5\pi}{6}\Rightarrow\theta=\frac{\pi}{12}$ and $\frac{5\pi}{12}$. The curve $r=4\sin(2\theta)$ is outer and $r=2$ is inner. Thus the area is

$A=4\int_{\pi/12}^{5\pi/12}\frac{1}{2}((4\sin(2\theta))^2-2^2)d\theta=2\int_{\pi/12}^{5\pi/12}(16\sin^2(2\theta)-4)d\theta=2\int_{\pi/12}^{5\pi/12}(8(1-\cos(4\theta))-4)d\theta=2\int_{\pi/12}^{5\pi/12}(4-8\cos(4\theta))d\theta=2(4\theta-\frac{8}{4}\sin(4\theta))|_{\pi/12}^{5\pi/12}=2(4\frac{\pi}{3}-2\sin\frac{5\pi}{3}+2\sin\frac{\pi}{3})=\frac{8\pi}{3}+4\sqrt{3}=15.306$

9. (a) $r=2\cos\theta\Rightarrow r'=-2\sin\theta$. Thus $L=\int_0^{\pi/2}\sqrt{4\cos^2\theta+4\sin^2\theta}d\theta=\int_0^{\pi/2}\sqrt{4}d\theta=\pi$.

(b) $r=e^{2\theta}\Rightarrow r'=2e^{2\theta}$. Thus $L=\int_0^{\pi/2}\sqrt{e^{4\theta}+4e^{4\theta}}d\theta=\int_0^{\pi/2}e^{2\theta}\sqrt{1+4}d\theta=\frac{\sqrt{5}}{2}e^{2\theta}|_0^{\pi/2}=\frac{\sqrt{5}}{2}(e^{\pi}-1)=24.75$.

(c) $r=\theta^2\Rightarrow r'=2\theta$. $L=\int_0^{2\pi}\sqrt{\theta^4+4\theta^2}d\theta=\int_0^{2\pi}\sqrt{\theta^2+4}\theta d\theta$. Using the substitution $\theta^2+4=u$ obtain $L=\frac{1}{3}(4\pi^2+4)^{3/2}-8)=92.896$.

(d) You can calculate the total length by finding the length of one petal and multiplying it by four. For the petal symmetrically around the $x$-axis, find the bounds from $r=\cos 2\theta=0\Rightarrow 2\theta=\pm\frac{\pi}{2}\Rightarrow\theta=\pm\frac{\pi}{4}$. Set up the integral for the length first (*before* you use the program). $L=4\int_{-\pi/4}^{\pi/4}\sqrt{\cos^2 2\theta+4\sin^2 2\theta}d\theta$. Switch your calculator back to the function mode. With $Y_1=4\sqrt{\cos^2 2x+4\sin^2 2x}$ and $n=100$ obtain the length of $L=9.69$.

(e) Find the total length as the length of one petal multiplied by three. Find the bounds from $r=\sin 3\theta=0\Rightarrow 3\theta=0$ and $3\theta=\pi\Rightarrow\theta=0$ and $\theta=\frac{\pi}{3}$. Then $L=3\int_0^{\pi/3}\sqrt{\sin^2 3\theta+9\cos^2 3\theta}d\theta$. Switch your calculator back to function mode and enter $Y_1=3\sqrt{\sin^2 3x+9\cos^2 3x}$. Obtain that $L=6.68$.