Calculus 2
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Review for Exam 1

a) **Integrals.** Evaluate the following integrals.

1. $\int \frac{x}{\sqrt{1-9x^2}} \ dx$
2. $\int \frac{1}{\sqrt{1-9x^2}} \ dx$
3. $\int \frac{x-1}{x^2} \ dx$
4. $\int xe^{x^2+1} \ dx$
5. $\int 2^{3x+1} \ dx$
6. $\int x\sqrt{x-3} \ dx$
7. $\int x^3\sqrt{1+x^2} \ dx$
8. $\int bxe^{ax^2+1} \ dx$ where a and b are arbitrary constants.
9. $\int \frac{1}{ax+b} \ dx$ where a and b are arbitrary constants.
10. $\int \frac{1}{4x^2+1} \ dx$
11. $\int \frac{1}{x^2+4} \ dx$
12. $\int \frac{x+1}{x^2+4} \ dx$
13. $\int (e^{2x} + e^{-2x}) \ dx$
14. $\int \cos(5x+1) \ dx$
15. $\int (2+\sin 2x) \ dx$
16. $\int \frac{1}{9x^2+1} \ dx$
17. Find the function $f(x)$ which has the derivative $f'(x) = \frac{10}{\sqrt{4x+1}}$ and satisfies the condition $f(0) = 3$.
18. Find the function $f(x)$ which has the derivative $f'(x) = x^2\sqrt{4x^4+9x^2}$ and and satisfies the condition $f(0) = -2$.

b) **Derivatives**. Find the derivative:

1. $y = 2^{\sin x}$
2. $y = x^{\sin x}$
3. $y = 3^{x^2+3x}$
4. $y = xe^{x^2+1}$
5. $y = (5x)^{\ln x}$
6. $y = \log_3(x^2+1)$
7. $y = ax\ln(x^2+b^2)$ where a and b are arbitrary constants.
8. $y = \frac{x\ln x}{x^2+1}$
9. $y = (3x+2)^{2x-1}$
10. $y = \sin(3x+2)\cos(2x-3)$
11. $y = \sin^{-1}(2x)$
12. $y = x\tan^{-1}\sqrt{x}$
13. $y = \sin^{-1} x^2 + \sqrt{1-x^2}$

c) **Equations**. Solve the following equations for x. In parts 7. – 9., find all solutions on interval $[0, 2\pi]$.

1. $2^{x-1} = 5$
2. $\log_3(x+4) = 1$
3. $3^{2x+3} = 7$
4. $\log_5(x^2+9) = 2$
5. $e^{4x-8} = 1$
6. $\ln(x+2) + \ln e^3 = 7$
7. $\sin x = \frac{2}{5}$
8. $\cos^2 x = \frac{1}{4}$
9. $\sin x \cos x = \sin x$

d) **Approximate Integration.** Approximate the following integral using the Left and Right Sums Program to first two nonzero digits.

1. $\displaystyle\int_0^2 \ln(x^2+1)\,dx$
2. $\displaystyle\int_1^3 \frac{e^{2x}}{x}\,dx$

e) **Area.** Find the following areas.

1. Area between $f(x) = x^2 - 2x$ and x-axis for $1 < x < 3$
2. Area between $f(x) = 2\sqrt{x} - 4$ and x-axis for $0 < x < 9$
3. Area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ for $1/2 < x < 1$
4. Area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ for $1 < x < 4$
5. Area between $y = 4 - x^2$ and $y = -x + 2$
6. Area between $y = 2x$ and $y = x^2 - 4x$
7. Area between $y = 4x^2$ and $y = x^2 + 3$
8. Area between $y = x$, $y = 2x$, and $y = 6 - x$
9. Area between $y^2 = x$, and $x - 2y = 3$
10. Area between $y = x^3$ and $y = 3x^2 - 2x$
11. Area between $y = x^3$ and $y = x$
12. Area between $y = -x^3$ and $y = 12x - 7x^2$

f) **Volume.** Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

1. $y = x^2$, $y^2 = x$ about x-axis.
2. $y = x^2$, $y = 2x$ about x-axis.
3. $x + y = 2$, $y = x^2$, $x = 0$ about y-axis.
4. $y = x^2 + 4$, $y = 6x - x^2$ about x-axis.
5. $y = x^2 + 4$, $y = 6x - x^2$ about y-axis.
6. $y = 2x^2 + 3$, $y = x^2 + 4$ about x-axis.
7. $y = 2x^2 + 3$, $y = x^2 + 4$, $x > 0$, about y-axis.
8. $y = -1 - 2x^2$ and $y = -5 - x^2$ about x-axis.
9. $y = -x^3$ and $y = 12x - 7x^2$ about y-axis.

g) **L'Hôpital's Rule.** Evaluate the limits:

1. $\displaystyle\lim_{x\to 0} \frac{e^{4x} - 1}{\sin 2x}$
2. $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} 2x}{x}$
3. $\displaystyle\lim_{x\to 0} \frac{1 - \cos 3x}{x^2}$
4. $\displaystyle\lim_{x\to 0} (1 + 3x)^{1/x}$
5. $\displaystyle\lim_{x\to \infty} (1 - \frac{5}{x})^{2x}$
6. $\displaystyle\lim_{x\to \infty} \ln(x+2) - \ln(x-1)$

h) **Average value. Average rate of change.**

1. Find the average value of $f(x) = 4 - x^2$ on $[0, 2]$. Find the x-value that corresponds to the average value f_{ave}.

2. Consider $f(x) = \ln(x^2+1)$ on interval [0,2]. (a) Use the Left-Right Sums program to calculate the average value of $f(x)$ on [0,2] to first two nonzero digits.
   (b) Find the average rate of change of $f(x)$ on [0,2].

i) **Applications.**

1. A chemical reaction produces a compound X with a rate of 23, 19, 12, 11, 9, 5, 2 liters per second at time intervals spaced by 1 second. Approximate the total volume of the compound X produced in the 6 seconds for which the rate is given.
2. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{\sqrt{t^2+9}}$ where t is the time in seconds and v is the velocity in feet per second. (a) Determine by how much the object moved between 3 and 5 seconds. (b) Knowing that when $t = 4$ seconds, the position function $s(t) = 8$ feet, determine the position function $s(t)$.
3. The size of a certain bacteria culture grows at a rate of $f(t) = te^{t/2}$ milligrams per hour. Use your calculator program to approximate (a) the total increase in the bacteria size after the first 3 hours to first two nonzero digits; (b) the average rate at which the bacteria size is increasing during the first 3 hours to first two nonzero digits.
4. Approximate the area of the lake using the shown measurements of its width which were taken 50 feet apart.

0	88	110	145	180	138	129	93	84	0

5. The velocity of blood that flows in a blood vessel with radius R and length l at distance r from central axis is $v(r) = \frac{P}{4\eta l}(R^2 - r^2)$ where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity over the interval $[0, R]$.
6. A force of 40 Newtons is required to hold a spring that has been stretched from its natural length of 10cm to 15cm. Determine the amount of work done in stretching the spring from 15cm to 18cm. To find the spring constant correctly, convert the centimeters to meters.
7. Suppose that 2 Joules of work are needed to stretch a spring from its natural length of 30cm to a length of 42cm. (a) Determine how much work is needed to stretch it from 35cm to 40cm. To find the spring constant correctly, convert the centimeters to meters. (b) Determine how far beyond its natural length will a force of 30N keep the spring stretched.
8. The force of $F(x) = axe^{-bx^2}$ Newtons acts on an object located at a distance x meters away from the initial position and a and b are positive constants. Determine the work needed to move the object from the initial position to b meters away from it.
9. In a certain city the temperature (in F) t hours after 9 am was approximated by the function $T(t) = 50 + 14\sin\frac{\pi t}{12}$. Find the average temperature during the period from 9 am to 9 pm.
10. Breathing is cyclic and a full respiratory cycle takes about 5 seconds. To model the amount of air present in the lungs we can use the function $f(t) = \frac{1}{2}\sin\frac{2\pi t}{5}$. Find the average volume of inhaled air in the lungs in one respiratory cycle.

Review for Exam 1 – Solutions

More detailed solutions of the problems can be found on the class handouts.

a) Integrals. 1. $-\frac{1}{9}\sqrt{1-9x^2}+c$ 2. $\frac{1}{3}\sin^{-1}(3x)+c$ 3. $\ln|x|+\frac{1}{x}+c$ 4. $\frac{1}{2}e^{x^2+1}+c$ 5. $\frac{1}{3\ln 2}2^{3x+1}+c$ 6. Use $u=x-3$. The integral becomes $\int(u+3)\sqrt{u}du=\int(u^{3/2}+3u^{1/2})du=\frac{2}{5}u^{5/2}+3\frac{2}{3}u^{3/2}+c=\frac{2}{5}(x-3)^{5/2}+2(x-3)^{3/2}+c$.
7. $u=1+x^2$. The integral becomes $\frac{1}{2}\int x^2\sqrt{u}\,du$. Use the formula $u=x^2+1$ to express x^2 in terms of u as $x^2=u-1$. The integral becomes $\frac{1}{2}\int(u-1)\sqrt{u}du=\frac{1}{2}\int(u^{3/2}-u^{1/2})du=\frac{1}{2}(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2})+c=\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+c$. 8. $\frac{b}{2a}e^{ax^2+1}+c$. 9. $\frac{1}{a}\ln|ax+b|+c$. 10. $\frac{1}{2}\tan^{-1}(2x)+c$ 11. $\frac{1}{2}\tan^{-1}\frac{x}{2}+c$ 12. $\frac{1}{2}\ln(x^2+4)+\frac{1}{2}\tan^{-1}\frac{x}{2}+c$ 13. $\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+c$ 14. $\frac{1}{5}\sin(5x+1)+c$ 15. $2x-\frac{1}{2}\cos 2x+c$ 16. $\frac{1}{3}\tan^{-1}(3x)+c$ 17. $f(x)=5(4x+1)^{1/2}-2$ 18. $f(x)=\int x^2\sqrt{4x^4+9x^2}dx$. Use $u=4x^2+9$. $f(x)=\int x^3\sqrt{u}\frac{du}{8x}=\frac{1}{8}\int x^2\sqrt{u}\,du$. From $u=4x^2+9$, $x^2=\frac{1}{4}(u-9)$ Thus $f(x)=\frac{1}{32}\int(u-9)\sqrt{u}du=\frac{1}{32}\int(u^{3/2}-9u^{1/2})du=\frac{1}{32}(\frac{2}{5}u^{5/2}-9\frac{2}{3}u^{3/2})+c=\frac{1}{80}(4x^2+9)^{5/2}-\frac{3}{16}(4x^2+9)^{3/2}+c$. $f(0)=-2\Rightarrow c=\frac{1}{40}$ resulting in $f(x)=\frac{1}{80}(4x^2+9)^{5/2}-\frac{3}{16}(4x^2+9)^{3/2}+\frac{1}{40}$.

b) Derivatives 1. $2^{\sin x}\cos x\ \ln 2$ 2. Logarithmic differentiation. $y'=x^{\sin x}(\cos x\ln x+\frac{\sin x}{x})$ 3. $3^{x^2+3x}\ln 3(2x+3)$ 4. $e^{x^2+1}+2x^2e^{x^2+1}$ 5. Logarithmic differentiation. $(5x)^{\ln x}(\frac{\ln(5x)}{x}+\frac{\ln x}{x})$ 6. $\frac{1}{\ln 3}\cdot\frac{2x}{1+x^2}$ 7. $a\ln(x^2+b^2)+\frac{2ax^2}{x^2+b^2}$. 8. $\frac{(\ln x+1)(x^2+1)-2x^2\ln x}{(x^2+1)^2}$ 9. Logarithmic differentiation. $(2\ln(3x+2)+\frac{3(2x-1)}{3x+2})(3x+2)^{2x-1}$ 10.$3\cos(3x+2)\cos(2x-3)-2\sin(2x-3)\sin(3x+2)$ 11. $\frac{2}{\sqrt{1-4x^2}}$ 12. $\tan^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x}\frac{1}{1+x}$ 13. $\frac{2x}{\sqrt{1-x^4}}-\frac{x}{\sqrt{1-x^2}}$

c) Equations. 1. 3.32 2. -1 3. -.61 4. $x=\pm4$ 5. $x=2$ 6. $x=e^4-2$ 7. $x\approx .411$ and $x\approx 2.73$ radians or 23.57 and 156.42 degrees. 8. $x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ or 60, 120, 240, and 300 degrees. 9. $x=0$ and π or 0 and 180 degrees.

d) Approximate Integration. 1. With $n=100$, left sum = right sum =1.4 2. With $n=300$, left sum = right sum = 81

e) Area. 1. 2 2. 32/3 3. 2.61 4. 1 5. 9/2 6. 36 7. 4 8. 3 9. 32/3 10. 1/2 11. 1/2 12. 11.25+.583=11.83

f) Volume. 1. $3\pi/10$ 2. $64\pi/15$ 3. $5\pi/6$ 4. $13\pi/3$ 5.π 6. $152\pi/15$ 7. $\pi/2$ 8. 281.49 or $448\pi/5$ 9. 106.18 (requires 2 integrals).

g) L'Hôpital's Rule. 1. 2 2. 2 3. 9/2 4. e^3 5. e^{-10} 6. 0

h) Average value and average rate. 1. $f_{ave}=\frac{8}{3}$. $4-x^2=\frac{8}{3}\Rightarrow x=\pm1.15$. -1.15 is not in (0,2) so $x=1.15$ is the x-value which corresponds to f_{ave}. 2. (a) $f_{ave}\approx 0.72$. (b) $f'_{ave}=\frac{\ln 5}{2}\approx 0.8047$.

i) Applications. 1. Left =upper estimate = 79 liters, Right = lower estimate = 58 liters. 2. a) 1.59 ft b) $\sqrt{t^2+9}+3$ 3. (a) Want size $=\int_0^3$ rate $dt=\int_0^3 te^{t/2}dt$. With $n=100$, get Left=Right=13 milligrams. (b) Want $f_{ave}=\frac{1}{3-0}\int_0^3 f(t)\,dt=\frac{1}{3}\int_0^3 te^{t/2}dt$. With $n=300$, get Left=Right=4.3 milligrams. 4. Left = Right = 48350 square feet. 5. $v_{ave}=\frac{1}{R}\int_0^R\frac{P}{4\eta l}(R^2-r^2)dr=\frac{P}{4\eta lR}(R^2r-\frac{1}{3}r^3)|_0^R=\frac{P}{4\eta lR}\frac{2}{3}R^3=\frac{PR^2}{6\eta l}$. 6. $k=800$ and $W=\int_{0.05}^{0.08}800xdx=1.56$ Joules. 7.Use $2=\int_0^{0.12}kxdx$ to find $k=277.78$. (a) $W=\int_{0.05}^{0.1}277.78xdx=277.78\frac{x^2}{2}|_{0.05}^{0.1}$ =1.0417 J (b) $30=277.78x\Rightarrow x=\frac{30}{277.78}=0.108$ m= 10.8cm. 8. $W=\int_0^b F(x)dx=\int_0^b axe^{-bx^2}=\frac{a}{2b}(-e^{-b^3}+1)$ or $\frac{a}{2b}(1-e^{-b^3})$. 9. $\frac{1}{12}\int_0^{12}(50+14\sin\frac{\pi t}{12})dt=\frac{1}{\pi}(50\frac{\pi t}{12}-14\cos\frac{\pi t}{12}|_0^{12}=50+\frac{28}{\pi}\approx 58.9$. F 10. Just inhaled air $\Rightarrow$ bounds are 0 and $\frac{5}{2}$. The average volume is $\frac{2}{5}\int_0^{5/2}\frac{1}{2}\sin\frac{2\pi t}{5}dt=\frac{2}{5}\ \frac{1}{2}\ \frac{-5}{2\pi}\ \cos\frac{2\pi t}{5}|_0^{5/2}=\frac{-1}{2\pi}(-1-1)=\frac{1}{\pi}\approx 0.318$.