Calculus 2
Lia Vas

Review for Exam 3

a) **Solutions of differential equations.**

1. Check if $y = x^2$ and $y = 2+e^{-x^3}$ are solutions of differential equation $y'+3x^2y = 6x^2$.
2. Show that $y = ce^{2x}$ is a solution of differential equation $y'' - 6y' + 8y = 0$ for any constant c.
3. Show that $y = c_1 \cos 2x + c_2 \sin 2x$ is a solution of differential equation $y'' + 4y = 0$ for any value of the constants c_1 and c_2.
4. Find value of constants A, B and C for which the function $y = Ax^2 + Bx + C$ is the solution of the equation $y'' - y' + 4y = 8x^2$.
5. Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

b) **General Solution.** Find the general solutions of the following.

1. $y' = 3x^2y$
2. $y' = y^2xe^{2x}$
3. $xy' + 2y = \cos x$
4. $y' = x(y + 1)$
5. $xy' + 2y = x^3$

c) **Particular Solution.** Solve the following initial-value problems.

1. $y' = xy, \quad y(0) = 5$
2. $y' - 2y = x, \quad y(0) = 0$
3. $y' = \frac{xy}{x^2+1}, \quad y(0) = 2$
4. $y' + 2y = 2e^x, \quad y(0) = 1$
5. $y' = 3y\sqrt{5 - 2x}, \quad y(\frac{5}{2}) = 3$

d) **Autonomous Equations.** Find the equilibrium solution(s) of the following, check the stability and sketch the graph of all the solutions.

1. $y' = y^2 - 2y$
2. $y' = (y + 1)(y - 2)^2$
3. $y' = y(y + 1)(y - 2)$
4. $y' = y(2 - y)^2(5 - y)^3$

e) **Approximate Solutions**

1. Use Euler's method with the step size 0.1 to approximate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = x + y, \quad y(0) = 1$. Sketch the solution.
2. Use Euler's method with the step size 0.2 to approximate $y(2)$ where $y(x)$ is the solution of the initial-value problem $y' = y - e^{-x}, \quad y(0) = 1$. Sketch the solution.
3. Use Euler's method with the step size 0.1 to approximate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = \sin(x + y), \quad y(0) = 0$. Sketch the solution.

f) **Differential Equations – Applications.**

1. A population of bacteria grows at a rate proportional to the size of population. The proportionality constant is 0.7. Initially, the population consist of two members. Find the population size after six days.
2. The size of a population of rabbits is modeled by differential equation $P' = -kP(100-P)$ where k is a positive parameter. Estimate the size of the population after a long period of time if the initial size of the population is 103 rabbits. Estimate the size of the population after a long period of time if the initial size of the population is 99 rabbits.
3. The Pacific halibut fishery is modeled by differential equation $B' = kB(K - B)$ where B is the biomass (total mass of the members of the population) in kilograms at time t, $K = 8 \cdot 10^7$ kg and $k = 8.7 \cdot 10^{-9}$ per year. Estimate the biomass after many years if the initial biomass is $9 \cdot 10^7$. Estimate the biomass after many years if the initial biomass is $3 \cdot 10^6$.
4. A glucose solution is administered intravenously into the bloodstream at a constant rate 4 mg/cm^3 per minute. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the concentration at that time with proportionality constant 2. If the initial concentration is 1 mg/cm^3, set up the differential equation that models this situation and solve it.
5. Let $A(t)$ be the area of tissue culture at time t (in days). Let the final area of the tissue when the growth is complete be 10 cm^2. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to $\sqrt{A}$. So, a reasonable model for the growth of tissue is obtained by assuming that the rate of growth is jointly proportional to $\sqrt{A}$ and $10 - A$.

   a) Formulate the differential equation that models this situation.

   b) If the proportionality constant is 1/4 and the area of tissue culture initially is 1 cm^2, use the Euler's method program to approximate the area of the culture after 3, 5 and 10 days using the step size 0.2 for each approximation. Sketch the graph of the solution.

   c) Edit the list (use STAT button, then Edit) you used to approximate the area after 10 days. Suppose that we can consider the growth complete if the area is within 0.01 cm^2 from 10 cm^2. Determine the approximate time at which we can consider the growth complete.

g) **Parametric Curves. General. Tangents.**

1. Let $x = 2 + 2\cos t,\ \ y = 2\sin t$ (a) Sketch the curve for $\pi \le t \le 2\pi$ and indicate the direction in which the curve is traced as the parameter increases. (b) Find the value of parameter t which corresponds to the point $(2, -2)$.
2. Let $x = t^2,\ \ y = 6 - 3t$. (a) Find the tangent line at (9, 15). (b) Eliminate the parameter to find a Cartesian equation of the curve.
3. Find an equation of the tangent to the curve $x = e^t,\ \ y = e^{-t}$ at the point corresponding to the value $t = 0$.
4. Find the points on the curve $x = \cos t,\ \ y = \cos t \sin t$ where tangent is horizontal and vertical. If the curve crosses itself, find the point of self-intersection and find the equations of the two tangents at that point.

h) **Parametric Curves. Area**

1. Find the area bounded by the loop of the curve $x = t^2,\ \ y = t^3 - 3t$.
2. Find the area bounded by the curve $x = \cos t,\ \ y = \cos t \sin t$.
3. Find the area bounded by the curve $x = \sin t,\ \ y = \cos^2 t \sin t$ and x-axis.

i) **Parametric Curves. Arc Length and Surface Area**

1. Find the length of the curve $x = 2 + 2\cos t,\ \ y = 2\sin t$ from (4, 0) to (0, 0).
2. Find the length of the loop of $x = t^2,\ \ y = 3t - t^3$. Use the Left-Right Sums program with 100 steps to approximate the integral.
3. Find the length of the curve $x = \ln t,\ \ y = e^{-t},\ \ 1 \le t \le 2$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two digits.
4. Find the area of the surface generated by revolving the curve $x = t^3,\ \ y = t^2,\ \ 0 \le t \le 1$, about x-axis.
5. Find the area of the surface generated by revolving the curve $x = t + t^3,\ \ y = t - \frac{1}{t^2},\ \ 1 \le t \le 2$, about y-axis. Average the Left and the Right Sums with 100 steps to approximate the value of the integral computing the surface area.

Review for Exam 3 – Solutions

<u>The class handouts contain more detailed solutions of these problems.</u>

a) 1. $y = x^2$ is not a solution and $y = 2 + e^{-x^3}$ is a solution of the given equation.

2. Substitute $y' = 2ce^{2x}$, and $y'' = 4ce^{2x}$ into the equation and show that it simplifies to the identity $0 = 0$.

3. Substitute $y = c_1 \cos 2x + c_2 \sin 2x$ and its second derivative $y'' = -4c_1 \cos 2x - 4c_2 \sin 2x$ into the equation and note that it simplifies to the identity 0=0.

4. Find the derivatives of $y = Ax^2 + Bx + C$ to be $y' = 2Ax + B$ and $y'' = 2A$ and plug them into the equation $y'' - y' + 4y = 8x^2$. Equating the coefficients of polynomials on both sides of the equation obtain that $A = 2$, $B = 1$ and $C = \frac{-3}{4}$ so that $y = 2x^2 + x - \frac{3}{4}$ is a solution of differential equation.

5. Finding derivatives of $y = Ae^{3x}$, $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$, and substituting them into the equation produces the value $A = 3$. Thus, $y = 3e^{3x}$ is a solution of differential equation.

b) **General Solution.** 1. $y = e^{x^3+c} = e^{x^3}e^c$ or $y = Ce^{x^3}$. 2. $y = \frac{1}{\frac{-1}{2}xe^{2x}+\frac{1}{4}e^{2x}+c}$ 3. $y = \frac{1}{x}\sin x + \frac{1}{x^2}\cos x + \frac{c}{x^2}$ 4. $y = ce^{x^2/2} - 1$ 5. $y = \frac{x^3}{5} + \frac{c}{x^2}$

c) **Particular Solution.** 1. $y = 5e^{x^2/2}$ 2. $y = \frac{-x}{2} - \frac{1}{4} + \frac{1}{4}e^{2x}$ 3. $y = e^{1/2\ln(x^2+1)+\ln 2} = 2\sqrt{x^2+1}$ 4. $y = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}$ 5. $y = 3e^{-\sqrt{(5-2x)^3}}$

d) **Autonomous Equations.** 1. $y = 0$ is asymptotically stable, $y = 2$ is unstable. 2. $y = 2$ is semistable and $y = -1$ is unstable. 3. $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable. 4. $y = 0$ is unstable, $y = 2$ is semistable, and $y = 5$ is stable.

e) **Approximate Solutions.** 1. $y(1) \approx 3.187$ 2. $y(2) \approx 3.014$ 3. $y(1) \approx .501$

f) **Differential Equations – Applications.** 1. 133 bacteria 2. If $P(0) = 103$, $\lim_{t\to\infty} P = \infty$. So, the population size increases without bounds. If $P(0) = 99$, $\lim_{t\to\infty} P = 0$. So, the population size decreases to 0 in this case. 3. Both with $B(0) = 9 \cdot 10^7$ and with $B(0) = 3 \cdot 10^7$ the biomass eventually becomes $8 \cdot 10^7$ kg.

4. Equation $C' = 4 - 2C$, $C(0) = 1$. Solution $C(t) = 2 - e^{-2t}$.

5. a) Equation $dA/dt = k\sqrt{A}(10 - A)$ b) After 3 days, the area is 8.3 cm^2. After 5 days, the area is 9.67 cm^2. After 10 days, the area is 9.995 cm^2. c) After 9.2 days.

g) **Parametric Curves. General. Tangents.**

1. (a) Lower half of the circle of radius 2, centered at (2,0) traversed in counter-clockwise direction. (b) $(x, y) = (2, -2) \Rightarrow 2 + \cos t = 2,\ 2\sin t = -2 \Rightarrow \cos t = 0, \sin t = -1 \Rightarrow t = \frac{-\pi}{2}$ or $t = \frac{3\pi}{2}$.
2. (a) $y = \frac{1}{2}x + \frac{21}{2}$ (b) $y = 6 \mp 3\sqrt{x}$
3. $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = \frac{-1}{e^{2t}}$. At $t = 0$, the slope is $\frac{-1}{e^0} = -1$ and $(x, y) = (1, 1)$. The tangent is $y - 1 = -1(x - 1) \Rightarrow y = -x + 2$.
4. Horizontal tangents $y = \pm\frac{1}{2}$ at points $(\frac{\sqrt{2}}{2}, \frac{1}{2})$, $(\frac{-\sqrt{2}}{2}, \frac{-1}{2})$, $(\frac{-\sqrt{2}}{2}, \frac{1}{2})$, and $(\frac{\sqrt{2}}{2}, \frac{-1}{2})$, where t is $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$ respectively. Vertical tangent at $x = \pm 1$ where t is 0 and π. Self-intersection (0,0) for $t = \frac{\pi}{2}$ and $t = \frac{-\pi}{2}$. Tangents at (0,0) are $y = x$ and $y = -x$.

h) **Parametric Curves. Area.** 1. 8.31 2. $\frac{4}{3}$ 3. $\frac{1}{2}$

i) **Parametric Curves. Arc Length and Surface Area.** 1. 2π 2. 10.74 3. .73 4. 4.936 5. 307.5