

## Taylor Polynomials

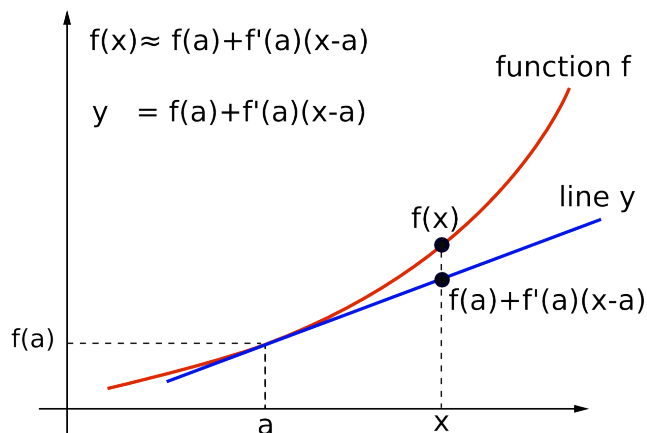
Recall that the line which approximates a function  $f(x)$  at a point  $(a, f(a))$  has the slope  $f'(a)$ . By point-slope equation, the equation of this line is

$$y - f(a) = f'(a)(x - a) \Rightarrow y = f(a) + f'(a)(x - a).$$

The expression  $f(a) + f'(a)(x - a)$  is called the **linear approximation** of  $f(x)$  at  $x = a$ .

$$f(x) \approx f(a) + f'(a)(x - a)$$

Note that the **function value and the value of the first derivative** is the same for a function and its linear approximation.



In applications, you can think of the value  $f(a)$  as of the **present value**, the value  $f(x)$  then represents the **future value**,  $(x - a)$  the **time lapsed** and  $f'(a)$  the **change rate**. Thus

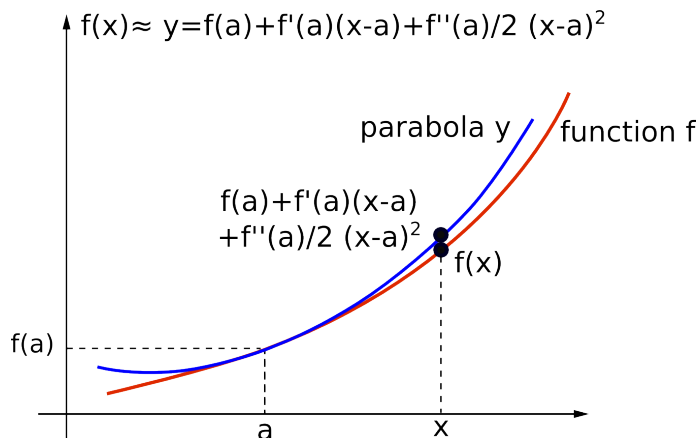
$f(x)$	$\approx$	$f(a)$	+	$f'(a)$	$(x - a)$
future value	$\approx$	present value	+	change rate	time elapsed

Assume now that we want to increase the accuracy of the approximation by approximating the function with a parabola  $y = A(x - a)^2 + B(x - a) + C$  in such a way that the **function value and the value of the first and the second derivatives** are the same for  $f(x)$  and parabola  $y$  when  $x = a$ . The condition  $f(a) = y(a)$  implies that  $C = f(a)$ . The condition that  $f'(a) = y'(a)$  implies that  $f'(a) = 2A(x - a) + B|_{x=a} = 0 + B$  so that  $B = f'(a)$ .

The condition  $f''(a) = y''(a)$  implies that  $f''(a) = 2A \Rightarrow A = \frac{f''(a)}{2}$ .

This produces the formula for polynomial of second degree that approximates the function  $f(x)$  at  $x = a$ .

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$



Similarly, if we are to approximate the formula of a polynomial of third degree  $y = A(x - a)^3 + B(x - a)^2 + C(x - a) + D$  which approximates function  $f(x)$  at  $x = a$ , we obtain that  $D = f(a)$ ,  $C = f'(a)$ ,  $B = \frac{f''(a)}{2}$ , and, equating  $f'''(a)$  with  $y'''(a) \Rightarrow f'''(a) = 2 \cdot 3A = 6A$  so that  $A = \frac{f'''(a)}{6}$ .

Continuing in this way, we obtain the formula for approximating  $f(x)$  with a polynomial of  $n$ -th degree to be

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{1 \cdot 2 \cdot 3}(x - a)^3 + \dots + \frac{f^{(n)}(a)}{1 \cdot 2 \cdot \dots \cdot n}(x - a)^n$$

The polynomial on the right is called the **Taylor polynomial of  $f(x)$  at  $x = a$  of order  $n$** .

The product  $1 \cdot 2 \cdot \dots \cdot n$  is written shortly as  $n!$  and is called the **factoriel** of  $n$ . Thus  $1! = 1$ ,  $2! = 1 \cdot 2 = 2$ ,  $3! = 1 \cdot 2 \cdot 3 = 6$ ,  $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ ,  $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$  and so on.  $0!$  is defined to be 1. Thus,

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Since  $0! = 1$  and  $1! = 1$  the first two terms can be written as  $f(a) = \frac{f^{(0)}(a)}{0!}$  and  $f'(a)(x - a) = \frac{f'(a)}{1!}(x - a)$  so that the Taylor polynomial of degree  $n$  can be written as  $\frac{f^{(0)}(a)}{0!} + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$ . This last formula is often shortened as  $\sum_{i=0}^n \frac{f^{(i)}(a)}{i!}(x - a)^i$ .

Thus,

$$f(x) \approx \sum_{i=0}^n \frac{f^{(i)}(a)}{i!}(x - a)^i$$

The Taylor polynomial centered at 0 is sometimes called **Maclaurin polynomial**.

Approximating a function using its Taylor polynomial is particularly useful when certain phenomena is modeled by a function which is either

- too complex to be manipulated or
- such that its exact formulas is not known.

but its value and the value of its derivatives are known at a point. Calculators and software applications (including Matlab for example) manipulate many functions using their Taylor polynomials.

### Practice Problems.

1. Find the Taylor polynomial of the degree  $n$  centered at point  $a$  for function  $f(x)$ .

- |  |                                       |                                      |
|--|---------------------------------------|--------------------------------------|
| (a) $x^2 - 2x + 1$ ; $a = 0, n = 30$   | (b) $e^x$ ; $a = 0, n = 4$            | (c) $e^{2x}$ ; $a = 0, n = 4$        |
| (d) $xe^{2x}$ ; $a = 0, n = 5$         | (e) $\frac{1}{1-x}$ ; $a = 0, n = 4$  | (f) $\frac{1}{1+x}$ ; $a = 0, n = 4$ |
| (g) $\frac{x^2}{1+x}$ ; $a = 0, n = 6$ | (h) $\frac{1}{1-3x}$ ; $a = 0, n = 4$ | (i) $\sin x$ ; $a = 0, n = 4$        |

2. Find the Taylor polynomial of the given degree  $n$  centered at given point  $a$  for function  $f(x)$ .

- (a)  $f(x) = e^x$ ;  $a = 0$ ;  $n = 4$ . Use to approximate  $e$  with a rational number.
- (b)  $f(x) = e^x$ ;  $a = 1$ ;  $n = 4$ .
- (c)  $f(x) = \sin x$ ;  $a = 0$ ;  $n = 4$ . Use to approximate  $\sin(.2)$  with a rational number.
- (d)  $f(x) = e^x \sin x$ ;  $a = 0$ ;  $n = 3$ . Use to approximate  $e^{1/2} \sin \frac{1}{2}$  with a rational number.
3. If  $f(2) = 5$ ,  $f'(2) = 3$  and  $f''(2) = 1$ , approximate  $f(2.1)$ .
4. If  $f(2) = 5$ ,  $f'(2) = 3$ ,  $f''(2) = 1$ , and  $f'''(2) = \frac{1}{2}$  approximate  $f(1.9)$ .
5. If  $f(1) = f'(1) = -1$ ,  $f''(1) = f'''(1) = 0$  and  $f^{iv}(1) = 2$ , approximate  $f(1.01)$ .
6. (“PChem problem”) Approximate the function  $e^{\frac{hv}{kT}} - 1$  by its Taylor polynomial of the second degree in terms of  $v$ .
7. (“Physics problem”) The magnitude of the electric field  $E$  of a single charge  $q$  can be described by  $E = \frac{kq}{r^2}$  where  $r$  is the distance between the field and the charge and  $k$  is a proportionality constant. If two opposite charges on distance  $d$  from each other create an electric dipole moment, this formula changes to

$$E = \frac{kq}{(r-d)^2} - \frac{kq}{(r+d)^2} = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2}$$

Use the Taylor polynomial of the second degree of the function  $f(x) = \frac{1}{(1-x)^2}$  to show that the magnitude of the electric field  $E$  can be approximated as

$$E \approx \frac{4kqd}{r^3}$$

This approximation is accurate if  $r$  is much larger than  $d$  so that the quotient  $\frac{d}{r}$  is small.

### Solutions.

1. (a) Let  $f(x) = x^2 - 2x + 1$ . Then  $f(0) = 1$ ,  $f'(0) = -2$  and  $f''(0) = 2$ . All the other derivatives are 0. So, the Taylor polynomial is  $1 - 2x + \frac{2}{2}x^2 + 0 + 0 + \dots = 1 - 2x + x^2$ . Note that this is the same polynomial as  $f(x)$ .  
This answer should not be surprising. In fact, any polynomial is equal to its Taylor series expansion centered at 0.
- (b) Let  $f(x) = e^x$ . Then  $f^{(n)}(x) = e^x$  for any  $n$  and  $f^{(n)}(0) = 1$ . So,  $e^x \approx 1 + 1(x-0) + \frac{1}{2}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ .
- (c) To get the expansion for  $e^{2x}$ , substitute  $x$  with  $2x$  in the expansion for  $e^x$  from the previous problem. Thus  $e^{2x} \approx 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \frac{1}{24}(2x)^4 = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$ .
- (d) To get the expansion for  $xe^{2x}$ , multiply the expansion for  $e^{2x}$  by  $x$ . Obtain  $xe^{2x} \approx x + 2x^2 + 2x^3 + \frac{4}{3}x^4 + \frac{2}{3}x^5$ .
- (e)  $f(x) = \frac{1}{1-x} \Rightarrow f' = \frac{1}{(1-x)^2}$ ,  $f'' = \frac{2}{(1-x)^3}$ ,  $f''' = \frac{6}{(1-x)^4}$ ,  $f^{(4)} = \frac{24}{(1-x)^5}$ . At  $x = 0$ ,  $f(0) = 1$ ,  $f'(0) = 1$ ,  $f''(0) = 2$ ,  $f'''(0) = 6$ ,  $f^{(4)}(0) = 24$ . So  $\frac{1}{1-x} \approx 1 + 1(x-0) + \frac{2}{2}(x-0)^2 + \frac{6}{3!}(x-0)^3 + \frac{24}{4!}(x-0)^4 = 1 + x + x^2 + x^3 + x^4$ .

(f) To get the expansion for  $\frac{1}{1+x}$ , substitute  $-x$  for  $x$  in the expansion for  $\frac{1}{1-x}$ . Obtain  $1 - x + x^2 - x^3 + x^4$ .

(g) Multiply the polynomial in the previous problem by  $x^2$ . Obtain  $x^2 - x^3 + x^4 - x^5 + x^6$ .

(h) To get the expansion for  $\frac{1}{1-3x}$ , substitute  $3x$  for  $x$  in the expansion for  $\frac{1}{1-x}$ . Obtain  $1 + 3x + (3x)^2 + (3x)^3 + (3x)^4 = 1 + 3x + 9x^2 + 27x^3 + 81x^4$ .

(i)  $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x \Rightarrow f'''(x) = -\cos x \Rightarrow f^{(4)}(x) = \sin x$ .  
The values at 0 are: 0, 1, 0, -1, 0. So,  $\sin x \approx 0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 = x - \frac{1}{6}x^3$ .

2. (a) From problem 1 (b), we have that  $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ . Substitute 1 for  $x$  to approximate  $e = e^1$ . Obtain  $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{24+24+12+4+1}{24} = \frac{65}{24} = 2.7083$ . Comparing with the calculator answer  $e \approx 2.7183$ , you can see that just first five terms compute the first two digits correctly.

(b)  $e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4$

(c) From problem 1 (i) we have that  $\sin x \approx x - \frac{x^3}{6}$ . Substitute  $0.2 = \frac{1}{5}$  for  $x$  to approximate  $\sin \frac{1}{5}$ . Obtain  $\frac{1}{5} - \frac{1}{750} = \frac{149}{750} \approx .198666\dots$  Comparing with the calculator answer  $\sin \frac{1}{5} \approx .198669$ , you can see that the first five decimals are correct.

(d) Let  $f(x) = e^x \sin x$ . Find the first three derivatives  $f'(x) = e^x \sin x + e^x \cos x \Rightarrow f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x \Rightarrow f'''(x) = 2e^x \cos x - 2e^x \sin x$ . Plug 0 and get  $f(0) = 0$ ,  $f'(0) = 1$ ,  $f''(0) = 2$  and  $f'''(0) = 2$ . Thus  $e^x \sin x \approx 0 + 1x + \frac{2}{2}x^2 + \frac{2}{6}x^3 = x + x^2 + \frac{x^3}{3}$ .

To approximate  $e^{1/2} \sin \frac{1}{2}$  substitute  $\frac{1}{2}$  for  $x$  in the Taylor polynomial. Obtain  $\frac{1}{2} + \frac{1}{4} + \frac{1}{24} = \frac{19}{24} \approx .7917$ . Compare with the calculator answer  $e^{1/2} \sin \frac{1}{2} \approx .7904$ .

3. Let  $x = 2.1$  and  $a = 2$ . The Taylor polynomial of degree 2 gives you  $f(2.1) \approx f(2) + f'(2)(2.1 - 2) + \frac{f''(2)}{2}(2.1 - 2)^2 = 5 + 3(.1) + \frac{1}{2}(.1)^2 = 5.305$ .

4. Let  $x = 1.9$  and  $a = 2$ . Then  $f(1.9) \approx f(2) + f'(2)(1.9 - 2) + \frac{f''(2)}{2}(1.9 - 2)^2 + \frac{f'''(2)}{6}(1.9 - 2)^3 = 5 + 3(-.1) + \frac{1}{2}(-.1)^2 + \frac{1}{12}(-.1)^3 = 4.705$ .

5. Let  $x = 1.01$  and  $a = 1$ . Then  $f(1.01) \approx f(1) + f'(1)(1.01 - 1) + \frac{f''(1)}{2}(1.01 - 1)^2 + \frac{f'''(1)}{6}(1.01 - 1)^3 + \frac{f^{(4)}(1)}{24}(1.01 - 1)^4 = -1 - 1(.01) + \frac{2}{24}(.01)^4 = -1.00999 \approx -1.01$ .

6. The problem is asking you to find the second order Taylor polynomial centered at  $v = 0$ . Let  $f(v) = e^{\frac{hv}{kT}} - 1$ . Then  $f'(v) = \frac{h}{kT} e^{\frac{hv}{kT}}$  and  $f''(v) = \frac{h^2}{k^2 T^2} e^{\frac{hv}{kT}}$ . Thus  $f(0) = 1 - 1 = 0$ ,  $f'(0) = \frac{h}{kT}$ , and  $f''(0) = \frac{h^2}{k^2 T^2}$ . So  $f(v) \approx \frac{hv}{kT} + \frac{h^2 v^2}{2k^2 T^2} = \frac{hv(2kT + hv)}{2k^2 T^2}$ .

7. For  $f(x) = \frac{1}{(1-x)^2}$ ,  $f'(x) = \frac{2}{(1-x)^3}$  and  $f''(x) = \frac{6}{(1-x)^4}$ .  $f(0) = 1$ ,  $f'(0) = 2$  and  $f''(0) = 6$ . Thus, the second degree approximation is  $\frac{1}{(1-x)^2} \approx 1 + 2x + \frac{6}{2}x^2 = 1 + 2x + 3x^2$ .

Substituting  $x$  with  $-x$  we obtain that  $\frac{1}{(1+x)^2} = \frac{1}{(1-(-x))^2} \approx 1 + 2(-x) + 3(-x)^2 = 1 - 2x + 3x^2$ .

By treating  $\frac{d}{r}$  as  $x$  for the expansion of the terms  $\frac{1}{(1-\frac{d}{r})^2}$  and  $\frac{1}{(1+\frac{d}{r})^2}$ , we obtain that  $E = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2} \approx \frac{kq}{r^2} \left( 1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - \left( 1 - 2\frac{d}{r} + 3\frac{d^2}{r^2} \right) \right) = \frac{kq}{r^2} \left( 1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - 1 + 2\frac{d}{r} - 3\frac{d^2}{r^2} \right) = \frac{kq}{r^2} \left( 4\frac{d}{r} \right) = \frac{4kqd}{r^3}$ .