**Calculus 2**
**Lia Vas**

# Approximate Integration. Trapezoidal and Simpson's sums.

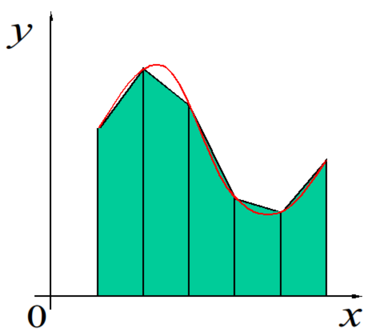
Recall that the Left and the Right Sums approximate the area under a curve as the sum of certain rectangles. On each subinterval, the left sum uses rectangles whose heights are obtained using the value at the left endpoints and the right sum uses rectangles whose heights are obtained using the values at the right endpoints. The **Trapezoidal Sum** uses trapezes whose upper side is obtained by connecting the left and the right endpoints.

Recall that the area of a trapeze with base $h$ and the heights $y_1$ and $y_2$ is given by $h\frac{y_1+y_2}{2}$. To approximate the area under $f(x)$ for $a \leq x \leq b$ using the Trapezoidal Sum with $n$ subintervals, the base of each trapeze is $h = \frac{b-a}{n}$. The first trapeze has the heights $f(x_0)$ and $f(x_1)$, the second $f(x_1)$ and $f(x_n)$, and so on. The last has the heights $f(x_{n-1})$ and $f(x_n)$. So, the formula below computer the sum of the areas of all trapezes

$$T = \frac{b-a}{n} \left( \frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} \ldots + \frac{f(x_{n-2})+f(x_{n-1})}{2} + \frac{f(x_{n-1})+f(x_n)}{2} \right)$$

$$= \frac{b-a}{n} \left( \frac{f(x_0)}{2} + f(x_1) + f(x_2) \ldots + f(x_{n-2}) + f(x_{n-1}) + \frac{f(x_n)}{2} \right).$$

Factor $\frac{1}{2}$ to obtain the final formula:




**Trapezoidal Sum**

$$T = \frac{b-a}{2n} \left( f(x_0) + 2f(x_1) + 2f(x_2) \ldots \right.$$
$$\left. \ldots + 2f(x_{n-1}) + f(x_n) \right).$$

Note that regrouping the terms of the last formula as

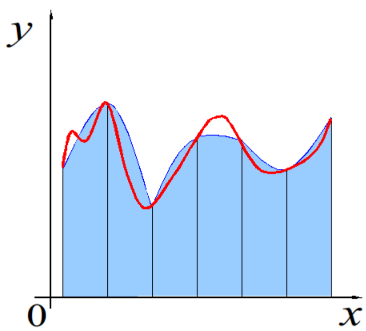
$$T = \frac{b-a}{2n} \left( f(x_0) + f(x_1) + \ldots + f(x_{n-1}) + f(x_1) + f(x_2) + \ldots + f(x_n) \right)$$

gives us

$$\frac{1}{2}\left( \frac{b-a}{n}(f(x_0) + f(x_1) + \ldots f(x_{n-1})) + \frac{b-a}{n}(f(x_1) + f(x_2) + \ldots + f(x_n)) \right) =$$
$$\frac{1}{2}(\text{Left Sum} + \text{Right Sum}).$$

We have seen that the Trapezoidal Sum uses a line connecting $f(x_i)$ and $f(x_{i+1})$ on the subinterval $[x_i, x_{i+1}]$. Thus, for Trapezoidal sum approximations, on each subinterval the function is approximated by a **line**. If **parabola** is used instead of a line, one obtains **the Simpson's Sum** approximation. Since *three* points are needed to determine a parabola, two subintervals with three endpoints are considered at a time. Because of this, the number of subintervals $n$ *has to be even* when using the Simpson's Sum.

The resulting formula can be obtained using calculating the formula for the area under a parabola (but we shall skip that part) and it turns out to be as follows:




**Simpson's Sum**

$$S = \frac{b-a}{3n}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots\right.$$
$$\left.\ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

It can be shown that the Simpson's Sum is more accurate that the Trapezoidal Sum and both of them are more accurate than the Left and the Right Sums.

**Calculator Programs**. The programs below calculate Trapezoidal Sum and Simpson's Sum of a given function on a given interval with the given number of subintervals. The input are the values of upper and lower bounds ($a$ and $b$) and the number of subintervals $n$. Also, the given function should be entered as $Y_1$. Recall that for Simpson's sum $n$ has to be even.

Entering the programs in your calculator is completely optional – you are **not required** to have them in your calculator.

```
PROGRAM: TRAPEZ
Disp ‘‘LOWER BOUND’’         (to display Disp, choose PRGM then I/0 menu)
Input A                      (to display Input, choose PRGM then I/0 menu)
Disp ‘‘UPPER BOUND’’
Input B
Disp ‘‘NUMBER OF SUBINTERVALS’’
Input N
0 → T
(B-A)/N → D
A → X
D*Y1/2→ T                    (to display Y1, choose VARS, Y-VARS, then 1: Function)
For (I, 1, N-1)              (to display For, choose PRGM then CTL menu)
A+D*I → X
T+D* Y1 → T
End                          (to display End, choose PRGM then CTL menu)
B → X
T +D*Y1/2 →T
```

```
Disp ''TRAPEZOIDAL SUM'', T
```

```
PROGRAM: SIMPSON
Disp ''LOWER BOUND''
Input A
Disp ''UPPER BOUND''
Input B
Disp ''EVEN NUMBER OF SUBINTERVALS''
Input N
0 → S
N/2 → K
(B-A)/N →D
For (I, 1, K)
A+2(I-1)*D →L
A+2I*D →R
(L+R)/2 →M
L → X
Y1 → L
R →X
Y1 → R
M → X
Y1 → M
S + (L+4M+R)*D/3 → S
End
Disp ''SIMPSON'S SUM'', S
```

You can use the following example to test if your programs are working. Consider $y = x^2$ on the interval [0,2]. To approximate the area under this curve using Trapezoidal and Simpson's Sums with 20 subintervals, enter $x^2$ as $Y_1$ in your calculator and execute the programs with $a = 0$, $b = 2$ and $n = 200$. The Trapezoidal Sum is 2.67 and the Simpson's Sum is $\frac{8}{3} = 2.666....$

Note that the Simpson's sum in this case produces the exact value of $\int_0^2 x^2 dx = \frac{x^3}{3}|_0^2 = \frac{8}{3}$. This should not be surprising since the function is a parabola to start with. So, the approximation with parabola produces the same function.

**Practice Problems.**

1. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 2 cm apart. The liver is 12 cm long and the cross-sectional areas, in square centimeters are 0, 58, 94, 106, 117, 63, 0. Use (a) Trapezoidal Sum, (b) Simpson's Sum to approximate the volume of the liver.

2. A chemical reaction produces a compound X with a rate of 23, 19, 12, 11, 9, 5, 2 liters per second at time intervals spaced by 1 second. Approximate the total volume of the compound X produced in the 6 seconds for which the rate is given using (a) Trapezoidal Sum, (b) Simpson's Sum.

3. **Computing the cardiac output.** The cardiac output of the heart is the volume of the blood pumped by the heart in unit time (the rate of flow into the aorta). The **dye dilution method** is used to measure the cardiac output. Dye is injected into the right atrium and flows into the heart. A probe measures the concentration of the dye leaving the heart in equally spaced times over a time interval $[0, T]$. Let $c(t)$ be the concentration of the dye, $F$ the rate of flow we want to determine and $A$ the amount of dye inserted. Then

$$F = \frac{A}{\int_0^T c(t)dt}$$

(a) The dye dilution method is used to measure the cardiac output with 8 mg of dye. The dye concentration (in milligrams per liter) is modeled by $c(t) = \frac{t}{4}(12 - t)$, $0 \leq t \leq 12$, where $t$ is measured in seconds. Find the cardiac output (in liters per seconds).

(b) A 5 mg bolus of dye is injected into the right atrium. The concentration of dye (mg/l) is measured in the aorta at one-second intervals as shown.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $c(t)$ | 0 | .4 | 2.8 | 6.5 | 9.8 | 8.9 | 6.1 | 4 | 2.3 | 1.1 | 0 |

Use the Simpson's Sum to estimate the cardiac output.

**Solutions.**

1. The volume can be found as the integral of the cross sections. Since the function of the cross-section at any point is not known, approximate integration can be used with $n = 6$ and

| $x$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $y$ | 0 | 58 | 94 | 106 | 117 | 63 | 0 |

. Thus, $a = 0$ and $b = 12$. (a) Trapezoidal Sum $= \frac{12-0}{2(6)}(0 + 2(58) + 2(94) + 2(106) + 2(117) + 2(63) + 0) = 876$. Thus, the volume is approximately 867 cm$^3$. (b) Simpson's Sum $= \frac{12-0}{3(6)}(0 + 4(58) + 2(94) + 4(106) + 2(117) + 4(63) + 0) = 886.67$. Thus, the volume is approximately 886.67 cm$^3$.

2.

| time (sec.) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| rate (l/sec.) | 23 | 19 | 12 | 11 | 9 | 5 | 2 |

Thus, $a = 0$, $b = 6$ and $n = 6$. (a) Trapezoidal Sum $= \frac{6-0}{2(6)}(23 + 2(19) + 2(12) + 2(11) + 2(9) + 2(5) + 2) = 68.5$. Thus, the volume is approximately 68.5 liters. (b) Simpson's Sum $= \frac{6-0}{3(6)}(23 + 4(19) + 2(12) + 4(11) + 2(9) + 4(5) + 2) = 69$. Thus, the volume is approximately 69 liters.

3. (a) $\int_0^{12} c(t)dt = \int_0^{12} \frac{t}{4}(12 - t)dt = \int_0^{12}(3t - \frac{t^2}{4})dt = (\frac{3t^2}{2} - \frac{t^3}{12})|_0^{12} = 72$ mg/liter sec. So the cardiac output is $F = \frac{8}{72} = \frac{1}{9}$. The units are $\frac{\text{mg}}{\text{mg/l sec}} = \frac{\text{l}}{\text{sec}}$. So, the cardiac output is .11 liters per second or 6.67 liters per minute.

(b) Approximate the integral $\int_0^{10} c(t)dt$ using the Simpson's Sum. It is $\frac{10-0}{3(10)}(0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9) + 2(6.1) + 4(4) + 2(2.3) + 4(1.1) + 0) = \frac{628}{15} = 41.87$. mg/l sec. So, the cardiac output is $F = \frac{5}{\frac{628}{15}} = \frac{75}{628}$ =.119 liters per second or 7.166 liters per minute.