

Volume - Washer method

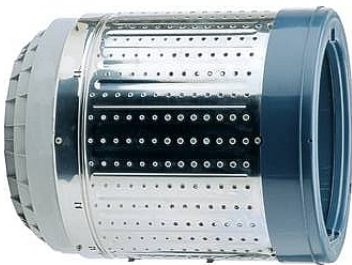
Computing general volume. If S is a solid between $x = a$ and $x = b$ with cross sectional area $A(x)$, then the volume V of S can be found by integrating the volume element dV which can be obtained as the product of $A(x)$ and x . Thus, the volume V can be found as

$$V = \int_a^b dV = \int_a^b A(x) dx.$$

Solids of revolution about x -axis: Let R be the region between the graph of $f(x)$ and x -axis on the interval $[a, b]$. If R is rotated about x -axis, the cross-section $A(x)$ is the area of the disk with radius $f(x)$.

$$A(x) = \pi(\text{radius})^2 = \pi(f(x))^2$$

So, the volume of the solid obtained by such revolution is given by



$$V = \int_a^b \pi(f(x))^2 dx.$$

Let R be the region between the graphs of $f(x)$ and $g(x)$ on the interval $[a, b]$ and let $f(x) \geq g(x)$. If R is rotated about x -axis, the cross-section is a washer with the area

$$A(x) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi(f(x))^2 - \pi(g(x))^2.$$

so, the volume of the solid obtained by such revolution is computed by

$$V = \int_a^b \pi(f(x))^2 - \pi(g(x))^2 dx.$$

Remarks:

1. If R is the region between $x = f(y)$ and $x = g(y)$, and we rotate R about y -axis, the volume of that surface of revolution is the same as if we were to rotate the region between $y = f(x)$ and $y = g(x)$ about x -axis.
2. Let R be the region between the graph of $f(x)$ and x -axis on the interval $[a, b]$. If R is rotated by an angle α (possibly not a full circle 2π), then the formula for the volume is

$$\text{Volume } V = \int_a^b \frac{\alpha}{2} (f(x))^2 dx.$$

3. Let R be the region between the graphs of $f(x)$ and $g(x)$ on the interval $[a, b]$ and let $f(x) \leq g(x) < 0$. When rotated about x -axis, note that $-f(x)$ and $-g(x)$ will be positive. Then $-f(x)$ determines the outer and $-g(x)$ the inner radius. For examples, see practice problems 8 and 9.

Practice Problems.

- a) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.
1. $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 2$ about the x -axis.
 2. $y = x^2$, $y = 4$, $x = 0$, about the x -axis.
 3. $y = x^2$, $y = 4$, about the x -axis.
 4. $y = x^2$, $y^2 = x$, about the x -axis.
 5. $y^2 = x$, $x = 2y$ about the y -axis.
 6. $y = x^2 + 4$, $y = 6x - x^2$ about x -axis.
 7. $y = 2x^2 + 2$, $y = x^2 + 6$ about x -axis.
 8. $y = -1 - 2x^2$ and $y = -5 - x^2$ about x -axis.
 9. (Extra credit level) $y = -x^3$ and $y = 12x - 7x^2$ about x -axis.
- b) Find the formula that computes the volume of:
1. Cylinder with height h and radius r .
 2. Sphere with radius r .
 3. Cone with height h and radius r .

Solutions.

- a)
1. $V = \int_1^2 \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^2 x^{-2} dx = -\pi x^{-1} \Big|_1^2 = \pi\left(\frac{-1}{2} + 1\right) = \frac{\pi}{2}$.
 2. Graph the curves first and determine the intersections. The curves $y = x^2$ and $y = 4$ intersect when $x^2 = 4 \Rightarrow x = \pm 2$. Since the region is bounded by $x = 0$ (y -axis), just one of the two intersections is relevant. So, the bounds of integration can be chosen to be 0 and 2 (note that 0, and -2 could work equally well). The curve $y = 4$ is greater than $y = x^2$ on $(0, 2)$ so the first is the outer and the second the inner radius. Hence, $V = \int_0^2 \pi(4^2 - (x^2)^2) dx = \pi \int_0^2 (16 - x^4) dx = (16x - \frac{x^5}{5}) \Big|_0^2 = \pi(32 - \frac{32}{5}) = \frac{128\pi}{5}$.
 3. Note that the condition $x = 0$ is removed from the previous problem. Thus, the bounds of integration are -2 and 2 (you are still integrating the same function). Obtain $V = \frac{256\pi}{5}$.
 4. If $y^2 = x$, then $y = \pm\sqrt{x}$ but just the positive branch intersect the curve $y = x^2$. Thus, you can consider $y = x^2$ and $y = \sqrt{x}$. The curves intersect when $x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = x(x^3 - 1) = 0 \Rightarrow x = 0$ and $x^3 = 1 \Rightarrow x = 1$ and $y = \sqrt{x}$ has larger values than $y = x^2$ on $(0, 1)$. Thus, the volume is $V = \int_0^1 \pi((\sqrt{x})^2 - (x^2)^2) dx = \pi \int_0^1 (x - x^4) dx = (\frac{x^2}{2} - \frac{x^5}{5}) \Big|_0^1 = \frac{3\pi}{10}$.

5. To use the washer method for this problem, interchange the variables x and y and consider rotating the region between $y = x^2$ and $y = 2x$ about the x -axis. The curves intersect when $x^2 = 2x \Rightarrow x(x - 2) = 0 \Rightarrow x = 0$ and $x = 2$ and the curve $y = 2x$ has greater values on $(0,2)$. Thus, $V = \int_0^2 \pi((2x)^2 - (x^2)^2)dx$ and this integral can be evaluated to equal $\frac{64\pi}{15}$.
6. Intersections: $x^2 + 4 = 6x - x^2 \Rightarrow 2x^2 - 6x + 4 = 2(x - 1)(x - 2) = 0 \Rightarrow x = 1, x = 2$. $y = 6x - x^2$ has larger values than $y = x^2 + 4$ on $(1,2)$ Be careful not to use 0 to determine which curve is upper since 0 is not in the interval $(1,2)$. $V = \int_1^2 \pi((6x - x^2)^2 - (x^2 + 4)^2)dx$. Square both terms and simplify before integrating term by term. The answer is $\frac{13\pi}{3}$.
7. Intersections: $2x^2 + 2 = x^2 + 6 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. $y = x^2 + 6$ is larger on $(-2,2)$. $V = \int_{-2}^2 \pi((x^2 + 6)^2 - (2x^2 + 2)^2)dx$. Square and simplify, then integrate. Obtain $V = \frac{1664\pi}{15}$.
8. Intersections $-1 - 2x^2 = -5 - x^2 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2$. Note that the curve $y = -5 - x^2$ is more negative than $y = -1 - 2x^2$. Thus, $y = -5 - x^2$ gives you the outer radius and $y = -1 - 2x^2$ the inner. So, the volume computes as $V = \int_{-2}^2 \pi((-5 - x^2) - (-1 - 2x^2))^2 dx$. This integral is equal to $\frac{448\pi}{5} = 281.49$.

Note that the same integral can be obtained when revolving $y = -(-5 - x^2 = 5 + x^2)$ and $y = -(-1 - 2x^2) = 1 + 2x^2$ about x -axis.

9. Intersections: $-x^3 = 12x - 7x^2 \Rightarrow 0 = x^3 - 7x^2 + 12x = x(x^2 - 7x + 12) = x(x - 3)(x - 4) \Rightarrow x = 0, x = 3$ and $x = 4$. On interval $(0,3)$, the curve $y = 12x - 7x^2$ is greater than $y = -x^3$. However, since the region in between the curves intersect the x -axis, you can compute the total volume over $(0,3)$ as sum of three integrals: the first as volume when revolving just $12x - 7x^2$ about x -axis on region before the x -intercept of $\frac{12}{7}$, the second when revolving just $-x^3$ over $(0, \frac{12}{7})$ and the third when revolving the area between the two curves over $(\frac{12}{7}, 3)$. Note that the outer radius is given by $y = -x^3$ in this case. Lastly, consider the interval $(3,4)$. The outer radius is given by $y = 12x - 7x^2$ (note that both curves are negative on $(3,4)$).

The total volume can be obtained as the sum of four integrals $V_1 = \int_0^{12/7} \pi(12x - 7x^2)^2 dx$, $V_2 = \int_0^{12/7} \pi(-x^3)^2 dx$, $V_3 = \int_{12/7}^3 \pi((-x^3)^2 - (12x - 7x^2)^2) dx$ and $V_4 = \int_3^4 \pi((12x - 7x^2)^2 - (-x^3)^2) dx$. These integrals compute to $\pi(24.18 + 6.216 + 54.99 + 51.657) = 430.53$.

- b)
 1. Represent the cylinder as the surface of revolution of horizontal line $y = r$ on interval $(0, h)$. $V = \int_0^h \pi r^2 dx$. Note that here you are integrating with respect to x , not r . Since r is a constant, you can factor it in front. So $V = \pi r^2 \int_0^h dx = \pi r^2 x \Big|_0^h = \pi r^2 h$ or $r^2 h \pi$. This represents the familiar formula for the product of area of the base $r^2 \pi$ and the height h .
 2. Represent the sphere as the surface of revolution of a circle of radius r . The formula for that circle is $x^2 + y^2 = r^2 \Rightarrow y = \pm \sqrt{r^2 - x^2}$. To get the whole sphere, it is sufficient to rotate the top part. Thus, you can consider just the positive root $y = \sqrt{r^2 - x^2}$. The bounds for the integration are $-r$ and r . Thus $V = \int_{-r}^r \pi(\sqrt{r^2 - x^2})^2 dx = \pi \int_{-r}^r (r^2 - x^2) dx = \pi(r^2 x - \frac{x^3}{3}) \Big|_{-r}^r = \pi(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}) = \frac{4r^3\pi}{3}$.
 3. Represent the cone as the surface of revolution of the line passing $(0,0)$ and (h,r) . The slope of this line is $\frac{r-0}{h-0} = \frac{r}{h}$. Since the y -intercept is 0, the equation is $y = \frac{r}{h}x$. The bounds are 0 and h so the volume is $V = \int_0^h \pi(\frac{r}{h}x)^2 dx = \pi \frac{r^2}{h^2} \frac{x^3}{3} \Big|_0^h = \pi \frac{r^2}{h^2} \frac{h^3}{3} = \frac{r^2 h \pi}{3}$.
Alternatively, you can consider revolving the line passing $(0,r)$ and $(h,0)$. This line has the equation $y = r - \frac{r}{h}x$.