Calculus 2
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Volume - Washer method

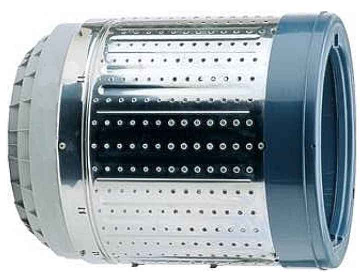
Computing general volume. If S is a solid between $x = a$ and $x = b$ with cross sectional area $A(x)$, then the volume V of S can be found by integrating the volume element dV which can be obtained as the product of $A(x)$ and dx. Thus, the volume V can be found as

$$V = \int_a^b dV = \int_a^b A(x)\ \ dx.$$

Solids of revolution about x-axis: Let R be the region between the graph of $f(x)$ and x-axis on the interval $[a, b]$. If R is rotated about x-axis, the cross-section $A(x)$ is the area of the disk with radius $f(x)$.

$$A(x) = \pi(\text{radius})^2 = \pi(f(x))^2$$

So, the volume of the solid obtained by such revolution is given by



$$\boxed{V = \int_a^b \pi(f(x))^2\ \ dx.}$$

Let R be the region between the graphs of $f(x)$ and $g(x)$ on the interval $[a, b]$ and let $f(x) \geq g(x)$. If R is rotated about x-axis, the cross-section is a washer with the area

$$A(x) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi(f(x))^2 - \pi(g(x))^2.$$

so, the volume of the solid obtained by such revolution is computed by

$$\boxed{V = \int_a^b \pi(f(x))^2 - \pi(g(x))^2\ \ dx.}$$

Remarks:

1. If R is the region between $x = f(y)$ and $x = g(y)$, and we rotate R about y-axis, the volume of that surface of revolution is the same as if we were to rotate the region between $y = f(x)$ and $y = g(x)$ about x-axis.

2. Let R be the region between the graph of $f(x)$ and x-axis on the interval $[a, b]$. If R is is rotated by an angle α (possibly not a full circle 2π), then the formula for the volume is

$$\text{Volume } V = \int_a^b \frac{\alpha}{2}(f(x))^2\ \ dx.$$

3. Let R be the region between the graphs of $f(x)$ and $g(x)$ on the interval $[a, b]$ and let $f(x) \leq g(x) < 0$. When rotated about x-axis, note that $-f(x)$ and $-g(x)$ will be positive. Then $-f(x)$ determines the outer and $-g(x)$ the inner radius. For examples, see practice problems 8 and 9.

Practice Problems.

a) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

1. $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 2$ about the x-axis.
2. $y = x^2$, $y = 4$, $x = 0$, about the x-axis.
3. $y = x^2$, $y = 4$, about the x-axis.
4. $y = x^2$, $y^2 = x$, about the x-axis.
5. $y^2 = x$, $x = 2y$ about the y-axis.
6. $y = x^2 + 4$, $y = 6x - x^2$ about x-axis.
7. $y = 2x^2 + 2$, $y = x^2 + 6$ about x-axis.
8. $y = -1 - 2x^2$ and $y = -5 - x^2$ about x-axis.
9. (Extra credit level) $y = -x^3$ and $y = 12x - 7x^2$ about x-axis.

b) Find the formula that computes the volume of:

1. Cylinder with height h and radius r.
2. Sphere with radius r.
3. Cone with height h and radius r.

Solutions.

a)
1. $V = \int_1^2 \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^2 x^{-2} dx = -\pi x^{-1}|_1^2 = \pi(\frac{-1}{2} + 1) = \frac{\pi}{2}$.
2. Graph the curves first and determine the intersections. The curves $y = x^2$ and $y = 4$ intersect when $x^2 = 4 \Rightarrow x = \pm 2$. Since the region is bounded by $x = 0$ (y-axis), just one of the two intersections is relevant. So, the bounds of integration can be chosen to be 0 and 2 (note that 0, and -2 could work equally well). The curve $y = 4$ is greater than $y = x^2$ on (0,2) so the first is the outer and the second the inner radius. Hence, $V = \int_0^2 \pi(4^2 - (x^2)^2)dx = \pi \int_0^2 (16 - x^4)dx = (16x - \frac{x^5}{5})|_0^2 = \pi(32 - \frac{32}{5}) = \frac{128\pi}{5}$.
3. Note that the condition $x = 0$ is removed from the previous problem. Thus, the bounds of integration are -2 and 2 (you are still integrating the same function). Obtain $V = \frac{256\pi}{5}$.
4. If $y^2 = x$, then $y = \pm\sqrt{x}$ but just the positive branch intersect the curve $y = x^2$. Thus, you can consider $y = x^2$ and $y = \sqrt{x}$. The curves intersect when $x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = x(x^3 - 1) = 0 \Rightarrow x = 0$ and $x^3 = 1 \Rightarrow x = 0$ and $x = 1$ and $y = \sqrt{x}$ has larger values than $y = x^2$ on (0,1). Thus, the volume is $V = \int_0^1 \pi((\sqrt{x})^2 - (x^2)^2)dx = \pi \int_0^1 (x - x^4)dx = (\frac{x^2}{2} - \frac{x^5}{5})|_0^1 = \frac{3\pi}{10}$.

5. To use the washer method for this problem, interchange the variables x and y and consider rotating the region between $y = x^2$ and $y = 2x$ about the x-axis. The curves intersect when $x^2 = 2x \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ and $x = 2$ and the curve $y = 2x$ has greater values on (0,2). Thus, $V = \int_0^2 \pi((2x)^2 - (x^2)^2)dx$ and this integral can be evaluated to equal $\frac{64\pi}{15}$.
6. Intersections: $x^2 + 4 = 6x - x^2 \Rightarrow 2x^2 - 6x + 4 = 2(x-1)(x-2) = 0 \Rightarrow x = 1, x = 2$. $y = 6x - x^2$ has larger values than $y = x^2 + 4$ on (1,2) Be careful not to use 0 to determine which curve is upper since 0 is not in the interval (1,2). $V = \int_1^2 \pi((6x - x^2)^2 - (x^2+4)^2)dx$. Square both terms and simplify before integrating term by term. The answer is $\frac{13\pi}{3}$.
7. Intersections: $2x^2 + 2 = x^2 + 6 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. $y = x^2 + 6$ is larger on $(-2, 2)$. $V = \int_{-2}^2 \pi((x^2+6)^2 - (2x^2+2)^2)dx$. Square and simplify, then integrate. Obtain $V = \frac{1664\pi}{15}$.
8. Intersections $-1 - 2x^2 = -5 - x^2 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2$. Note that the curve $y = -5 - x^2$ is more negative than $y = -1 - 2x^2$. Thus, $y = -5 - x^2$ gives you the outer radius and $y = -1 - 2x^2$ the inner. So, the volume computes as $V = \int_{-2}^2 \pi((-5-x^2) - (-1-2x^2)^2)dx$. This integral is equal to $\frac{448\pi}{5} = 281.49$.

   Note that the same integral can be obtained when revolving $y = -(-5 - x^2 = 5 + x^2)$ and $y = -(-1 - 2x^2) = 1 + 2x^2$ about x-axis.
9. Intersections: $-x^3 = 12x - 7x^2 \Rightarrow 0 = x^3 - 7x^2 + 12x = x(x^2 - 7x^2 + 12) = x(x-3)(x-4) \Rightarrow x = 0$, $x = 3$ and $x = 4$. On interval (0,3), the curve $y = 12x - 7x^2$ is greater than $y = -x^3$. However, since the region in between the curves intersect the x-axis, you can compute the total volume over (0,3) as sum of three integrals: the first as volume when revolving just $12x - 7x^2$ about x-axis on region before the x-intercept of $\frac{12}{7}$, the second when revolving just $-x^3$ over $(0, \frac{12}{7})$ and the third when revolving the area between the two curves over $(\frac{12}{7}, 3)$. Note that the outer radius is given by $y = -x^3$ in this case. Lastly, consider the interval (3,4). The outer radius is given by $y = 12x - 7x^2$ (note that both curves are negative on (3,4)).

   The total volume can be obtained as the sum of four integrals $V_1 = \int_0^{12/7} \pi(12x - 7x^2)^2dx$, $V_2 = \int_0^{12/7} \pi(-x^3)^2dx$, $V_3 = \int_{12/7}^3 \pi((-x^3)^2 - (12x - 7x^2)^2)dx$ and $V_4 = \int_3^4 \pi((12x - 7x^2)^2 - (-x^3)^2)dx$. These integrals compute to π(24.18+6.216+54.99+51.657)=430.53.

b)

1. Represent the cylinder as the surface of revolution of horizontal line $y = r$ on interval $(0, h)$. $V = \int_0^h \pi r^2 dx$. Note that here you are integrating with respect to x, not r. Since r is a constant, you can factor it in front. So $V = \pi r^2 \int_0^h dx = \pi r^2 x|_0^h = \pi r^2 h$ or $r^2 h\pi$. This represents the familiar formula for the product of area of the base $r^2\pi$ and the height h.
2. Represent the sphere as the surface of revolution of a circle of radius r. The formula for that circle is $x^2 + y^2 = r^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$. To get the whole sphere, it is sufficient to rotate the top part. Thus, you can consider just the positive root $y = \sqrt{r^2 - x^2}$. The bounds for the integration are $-r$ and r. Thus $V = \int_{-r}^r \pi(\sqrt{r^2 - x^2})^2dx = \pi\int_{-r}^r (r^2 - x^2)dx = \pi(r^2 x - \frac{x^3}{3})_{-r}^r = \pi(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}) = \frac{4r^3\pi}{3}$.
3. Represent the cone as the surface of revolution of the line passing $(0, 0)$ and (h, r). The slope of this line is $\frac{r-0}{h-0} = \frac{r}{h}$. Since the y-intercept is 0, the equation is $y = \frac{r}{h}x$. The bounds are 0 and h so the volume is $V = \int_0^h \pi(\frac{r}{h}x)^2dx = \pi\frac{r^2}{h^2}\frac{x^3}{3}|_0^h = \pi\frac{r^2}{h^2}\frac{h^3}{3} = \frac{r^2 h\pi}{3}$.

   Alternatively, you can consider revolving the line passing $(0, r)$ and $(h, 0)$. This line has the equation $y = r - \frac{r}{h}x$.