Calculus 3
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The Chain Rule

Recall the chain rule for functions of single variable:

$$\text{If } y = f(x) \text{ and } x = g(t), \text{ then } y'(t) = y'(x) \cdot x'(t) \quad \left(\text{ or } \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}\right).$$

The chain rule for function $z = f(x, y)$ with $x = g(t)$ and $y = h(t)$ is

$$z'(t) = z_x \cdot x'(t) + z_y \cdot y'(t) \;\text{ or }\; \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

The chain rule for function $z = f(x, y)$ with $x = g(s, t)$ and $y = h(s, t)$ is

$$z_s = z_x \cdot x_s + z_y \cdot y_s, \qquad z_t = z_x \cdot x_t + z_y \cdot y_t$$

or

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Implicit functions. In some cases, an implicit function $F(x, y, z) = 0$ cannot be solved for z. To find the derivatives z_x and z_y in these cases, use implicit differentiation. Differentiating the equation with respect to x we obtain $F_x + F_z z_x = 0$. Solving for z_x gives us $z_x = -\frac{F_x}{F_z}$. Differentiating the equation with respect to y we obtain $F_y + F_z z_y = 0$. Solving for z_y gives us $z_y = -\frac{F_y}{F_z}$. Thus, the partial derivatives z_x and z_y are given by:

$$z_x = -\frac{F_x}{F_z} \quad \text{and} \quad z_y = -\frac{F_y}{F_z}$$

Practice problems.

1. Find the indicated derivatives.

   (a) $z = 3x^2 + 2xy - 5y^2$, $x = 2 + t^2$, $y = 1 - t^3$; $z'(t)$ when $t = 0$.

   (b) $z = x \ln(x + 2y)$, $x = \cos t$, $y = \sin t$; $z'(t)$ when $t = 0$.

   (c) $z = e^x y + xy^2$, $x = st$, $y = s^2 + t^2$; z_s and z_t at $(1, 1)$.

   (d) $z = x^2 + xy$, $x = e^t \cos s$, $y = e^t \sin s$; z_s and z_t at $(\pi, 0)$.

   (e) $xy^2 + yz^2 + zx^2 = 3$; z_x and z_y at $(1, 1, 1)$.

   (f) $x - yz = \cos(x + y + z)$; z_x and z_y at $(0, 1, -1)$.

2. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is rising at a rate of 0.15 K/s. The pressure P, volume V and temperature T are related by the equation $PV = 8.31T$. Find the rate of change of the volume when the pressure is 20 kPa and temperature 320 K.

3. The number N of bacteria in a culture depends on temperature T and pressure P which depend on time t in minutes. Assume that 3 minutes after the experiment started, the pressure is increasing at a rate of 0.1 kPa/min and the temperature at a rate of 0.5 K/min. The number of bacteria changes at the rates of 3 bacteria per kPa and 5 bacteria per Kelvin. Find the rate at which the number of bacteria is increasing 3 minutes after the experiment started.

4. The number of flowers N in a closed environment depends on the amount of sunlight S that the flowers receive and the temperature T of the environment. Assume that the number of flowers changes at the rates $N_S = 2$ and $N_T = 4$. If the temperature depends on time as $T(t) = 85 - \frac{8}{1+t^2}$ and the amount of sunlight decreases on time as $S(t) = \frac{1}{t}$, find the rate of change of the flower population at time $t = 2$ days.

5. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$ where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. Determine how fast the temperature is rising on the bug's path after 3 seconds.

6. If the trajectory of an object is given by a parametric curve, its **speed** at a point can be found as **the length of the tangent vector** at that point. Assume that an object moves in space away from its initial position so that after t hours it is at $x = 2t$ and $y = 2t^2 - 1$ and $z = \sqrt{3t+1}$ miles from its initial position. Find the speed of that object 5 hours after it started moving.

7. An object moves in xy-plane so that after t hours it is at $x = x(t)$ and $y = y(t)$ miles from its initial position. Suppose that after 2 hours, the object has the velocity so that $x'(2) = 3$ and $y'(2) = 5$. Find the speed of the object after 2 hours.

Solutions.

1. (a) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (6x+2y)(2t) + (2x-10y)(-3t^2)$. When $t = 0$, $x = 2 + 0^2 = 2$ and $y = 1 - t^3 = 1 = 0 = 1$. Thus, $z'(0) = (12+2)(0) + (4-10)0 = 0$.

   (b) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\ln(x+2y) + \frac{x}{x+2y})(-\sin t) + \frac{2x}{x+2y}(\cos t)$. When $t = 0$, $x = \cos 0 = 1$ and $y = \sin 0 = 0$. Thus, $z'(0) = (\ln(1+0) + \frac{1}{1+0})(0) + \frac{2}{1+0}(1) = 0 + 2 = 2$.

   (c) $z_s = z_x x_s + z_y y_s = (e^x y + y^2)t + (e^x + 2xy)2s$, $z_t = z_x x_t + z_y y_t = (e^x y + y^2)s + (e^x + 2xy)2t$. When $s = 1$, $t = 1$, $x = 1(1) = 1$ and $y = 1^2 + 1^2 = 2$. Thus $z_s(1, 1) = (e^1 2 + 2^2)1 + (e^1 + 2(1)2)2 = 4e + 12$ and $z_t(1, 1) = (e^1 2 + 2^2)1 + (e^1 + 2(1)2)2 = 4e + 12$.

   (d) $z_s = z_x x_s + z_y y_s = (2x+y)(-e^t \sin s) + x(e^t \cos s)$, $z_t = z_x x_t + z_y y_t = (2x+y)(e^t \cos s) + x(e^t \sin s)$. When $s = \pi$ and $t = 0$, $x = e^0 \cos\pi = -1$ and $y = e^0 \sin\pi = 0$. Thus, $z_s(\pi, 0) = (-2)(0) + (-1)(-1) = 1$ and $z_t(\pi, 0) = (-2)(-1) + (-1)(0) = 2$.

   (e) Use the formulas for derivatives of implicit function. Let $F = xy^2 + yz^2 + zx^2 - 3$. Then $F_x = y^2 + 2xz$, $F_y = 2xy + z^2$, $F_z = 2yz + x^2$ and so $z_x = -\frac{F_x}{F_z} = -\frac{y^2+2xz}{2yz+x^2}$ and $z_y = -\frac{F_y}{F_z} = -\frac{2xy+z^2}{2yz+x^2}$. At $(1, 1, 1)$, $z_x = -\frac{1+2}{2+1} = -1$ and $z_y = -\frac{2+1}{2+1} = -1$.

   (f) Use the formulas for derivatives of implicit function. Let $F = x - yz - \cos(x+y+z)$. Then $F_x = 1 + \sin(x+y+z)$, $F_y = -z + \sin(x+y+z)$, $F_z = -y + \sin(x+y+z)$ and so $z_x = -\frac{F_x}{F_z} = -\frac{1+\sin(x+y+z)}{-y+\sin(x+y+z)}$ and $z_y = -\frac{F_y}{F_z} = -\frac{-z+\sin(x+y+z)}{-y+\sin(x+y+z)}$. At $(0, 1, -1)$, $z_x = -\frac{1+0}{-1+0} = 1$ and $z_y = -\frac{1+0}{-1+0} = 1$.

2. We are given the rates $\frac{dP}{dt} = 0.05$ kPa/s, $\frac{dT}{dt} = 0.15$ K/s at $P_0 = 20$ kPa and $T_0 = 320$ K. We need to find the rate $\frac{dV}{dt}$ at (P_0, T_0).

   Since the derivatives are with respect to P and T, treat V as the dependent variable and solve for V in terms of P and T to obtain the dependence formula $V = \frac{8.31T}{P}$. By the chain rule, $\frac{dV}{dt} = \frac{\partial V}{\partial P}\frac{dP}{dt} + \frac{\partial V}{\partial T}\frac{dT}{dt}$. From $V = \frac{8.31T}{P}$ we find that $\frac{\partial V}{\partial P} = \frac{-8.31T}{P^2}$ and that $\frac{\partial V}{\partial T} = \frac{8.31}{P}$. At $P_0 = 20$ and $T_0 = 320$, we calculate that $\frac{\partial V}{\partial P} = \frac{-8.31(320)}{400} = -6.648$ liters/kPa and that $\frac{\partial V}{\partial T} = \frac{8.31}{20} = 0.4155$ liters/K. Thus, $\frac{dV}{dt} = (-6.648)(0.05) + (0.4155)(0.15) = -.27$ liter per second.

3. We are given the rates $P'(3) = 0.1$, $T'(3) = 0.5$, $N_P = 3$, and $N_T = 5$. By the chain rule formula, $N'(t) = N_T T' + N_P P'$. Thus $N'(3) = 5 \cdot 0.5 + 3 \cdot 0.1 = 2.8$ bacteria/minute.

4. We are given the rates $N_S = 2$ and $N_T = 4$. The remaining two rates S' and T' needed for the chain rule formula $N' = N_S S' + N_T T'$ can be found from the formulas for S and T.

   $S = \frac{1}{t} \Rightarrow S' = \frac{-1}{t^2}$. At $t = 2$, $S' = \frac{-1}{4}$.

   $T = 85 - \frac{8}{1+t^2} \Rightarrow T' = \frac{16t}{(1+t^2)^2}$. At $t = 2$, $T' = \frac{32}{25}$.

   Thus, $N'(2) = N_S S' + N_T T' = 2\frac{-1}{4} + 4\frac{32}{25} = 4.62$ flowers/day.

5. We are given the rates $T_x(2,3) = 4$ and $T_y(2,3) = 3$ and the problem is asking for the rate $\frac{dT}{dt}$ at $t = 3$.

   Using chain rule, $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$. $x' = \frac{1}{2\sqrt{1+t}}$ and $y' = \frac{1}{3}$. At $t = 3$, $x' = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ and $y' = \frac{1}{3}$. When $t = 3$, $x = \sqrt{1+3} = 2$ and $y = 2 + 1 = 3$ so the point $(2,3)$ at which the derivatives T_x and T_y are given corresponds exactly to $t = 3$. Thus, $\frac{dT}{dt} = 4\frac{1}{4} + 3\frac{1}{3} = 1 + 1 = 2$ degrees Celsius per second.

6. $x' = 2$, $y' = 4t$ and $z' = \frac{3}{2\sqrt{3t+1}}$. When $t = 5$, $x' = 2$, $y' = 20$ and $z' = \frac{3}{8}$. Thus, the tangent vector is $\langle 2, 20, \frac{3}{8}\rangle$. The speed is the length $|\langle 2, 20, \frac{3}{8}\rangle| = \sqrt{2^2 + 20^2 + (\frac{3}{8})^2} = 20.1$ miles/hour.

7. At $t = 2$, $\langle x', y'\rangle = \langle 3, 5\rangle$. The speed = length of $\langle 3, 5\rangle = \sqrt{3^2 + 5^2} = 5.83$ miles/ hour.