Calculus 3
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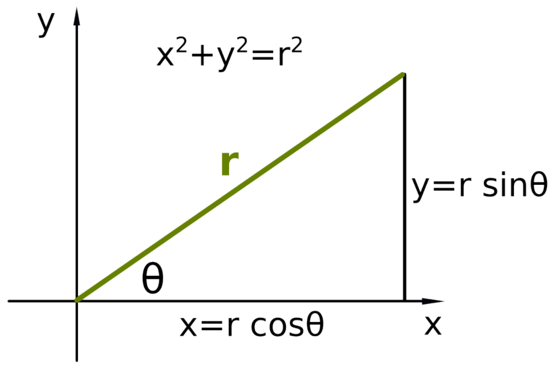
Double Integrals in Polar Coordinates
Volume of Regions Between Two Surfaces

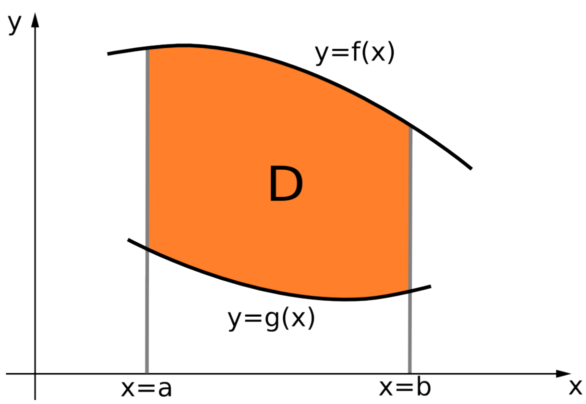
In many cases in applications of double integrals, the region in xy-plane has much easier representation in polar coordinates than in Cartesian, rectangular coordinates.

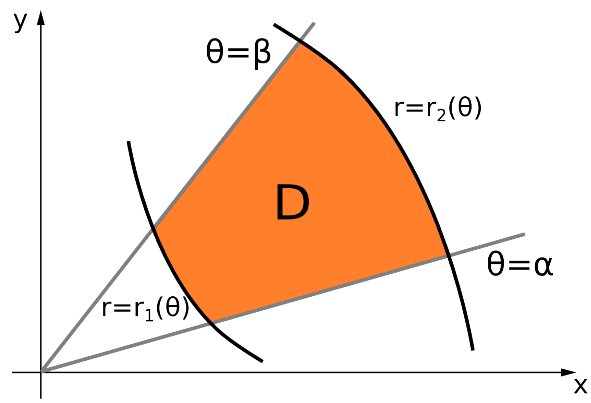
Recall that if r and θ are as in figure on the left, $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ so that



$$\boxed{x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad x^2 + y^2 = r^2.}$$

Recall also how the area between two curves given by functions of x on the first figure bellow corresponds to the area between two polar curves given by functions of θ.





Region between two curves in Cartesian and polar coordinates

Consider now a function $z = f(x, y)$ of two variables defined on a region D which can be represented in polar coordinates as follows.

$$D = \{\ (r, \theta) \mid \alpha \leq \theta \leq \beta, \ \ r_1(\theta) \leq r \leq r_2(\theta)\ \}.$$

The double integral of f over D is

$$\int\int_D f(x, y)\ dx\ dy = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$$

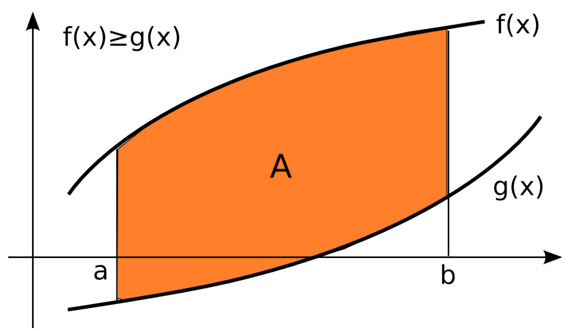
Note that $dxdy$ becomes $rdrd\theta$ in polar coordinates. The presence of r in this formula *will be explained in section on Parametric Surfaces.*

The use of polar coordinates can simplify evaluating the double integrals over regions bonded by circles or their parts. The bounds for r and θ in those cases can be determined in the same way as in Calculus 2.

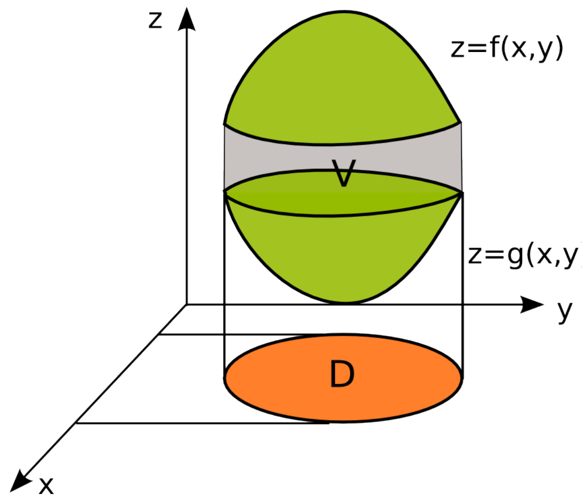
Area between curves in polar coordinates. Let D be a region in xy-plane which can be represented $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$ in polar coordinates. Using the formula for the area $A = \int\int_D dxdy$, we can demonstrate the validity of the formula for the area between polar curves from Calculus 2.

$$A = \int\int_D dxdy = \int\int_D r dr d\theta = \int_\alpha^\beta \left(\int_{r_1(\theta)}^{r_2(\theta)} r dr \right) d\theta = \int_\alpha^\beta \frac{1}{2} r^2 \Big|_{r_1(\theta)}^{r_2(\theta)} d\theta = \int_\alpha^\beta \frac{1}{2} \left((r_2(\theta))^2 - (r_1(\theta))^2 \right) d\theta$$

Volume of Regions Between Two Surfaces. Assume that two surfaces $z = f(x,y)$ and $z = g(x,y)$ are such that $f(x,y) \leq g(x,y)$ over a region D in xy plane. The volume between $f(x,y)$ and $g(x,y)$ over the region D can be found as the double integral of the difference $f(x,y) - g(x,y)$ over the region D. This can be shown using the same argument used in Calculus 1 when showing that the area of the region between two curves $f(x)$ and $g(x)$ such that $f(x) \geq g(x)$ on interval $[a,b]$ can be found by integrating the difference $f(x) - g(x)$ over the interval $[a,b]$.



Area $= \int_a^b$ (upper - lower curve) dx



Volume $= \int\int_D$ (upper - lower surface) dx

Practice problems.

1. Calculate the double integral

   a)
   $$\int\int_D x dx dy$$
   where D is the disk with center the origin and radius 5 in the first quadrant.

   b)
   $$\int\int_D xy dx dy$$
   where D is the region in the first quadrant between the circles $x^2+y^2 = 4$ and $x^2+y^2 = 25$.

   c)
   $$\int\int_D \frac{1}{\sqrt{x^2+y^2}} dx dy$$
   where D is the region inside the curve $r = 4\cos\theta$ and outside the curve $r = 2$.

d)

$$\int\int_D \frac{1}{\sqrt{x^2+y^2}}dxdy$$

where D is the region inside the curve $r = 2$ and outside the curve $r = 4\cos\theta$ in the first quadrant.

2. Find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$.

3. Find the volume of the solid inside the cylinder $x^2+y^2 = 4$ and between the cone $z = \sqrt{x^2+y^2}$ and the xy-plane.

4. Find the volume of the solid above the cone $z = \sqrt{x^2+y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$.

5. Find the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

6. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 2 - y$ and $z = 0$ in the first octant.

7. Using a double integral, find the area inside a loop of the four-leaved rose $r = \cos 2\theta$.

Solutions.

1. a) The bounds are $0 \leq r \leq 5$ and $0 \leq \theta \leq \frac{\pi}{2}$. Since $x = r\cos\theta$ and $dxdy = rdrd\theta$, the integral $\int\int_D xdxdy$ becomes $\int_0^{\pi/2}\int_0^5 r\cos\theta rdrd\theta = \int_0^{\pi/2}\cos\theta d\theta \ \int_0^5 r^2dr = \sin\frac{\pi}{2}\frac{r^3}{3}|_0^5 = \frac{125}{3}$

   b) $\int\int_D xydxdy = \int_0^{\pi/2}\int_2^5 r\cos\theta r\sin\theta rdrd\theta = \int_0^{\pi/2}\cos\theta\sin\theta d\theta \ \int_2^5 r^3dr = \frac{1}{2}(\frac{5^4}{4} - \frac{2^4}{4}) = \frac{609}{8} = 76.125$.

   c) Graph the region first. From the graph, you can see that the bounds for θ are determined by intersection of $r = 4\cos\theta$ and $r = 2$. Solving $4\cos\theta = 2$, yields $\cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$. The outer curve is $4\cos\theta$ and the inner is $r = 2$. The function $\frac{1}{\sqrt{x^2+y^2}}$ in polar coordinates is $\frac{1}{r}$. So, the integral $\int\int_D \frac{1}{\sqrt{x^2+y^2}}dxdy$ transforms to $\int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta}\frac{1}{r}rdrd\theta = \int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta}drd\theta = \int_{-\pi/3}^{\pi/3}(4\cos\theta - 2)d\theta = 4\sqrt{3} - 4\frac{\pi}{3}$

   d) Graph again the region first. From the graph, you can see that the bounds for θ are the intersection of the two curves and $\frac{\pi}{2}$. From part c), we have that the curves $r = 2$ and $r = 4\cos\theta$ intersect at $\frac{\pi}{3}$ in the first quadrant. So, $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$ and $4\cos\theta \leq r \leq 2$. The integral $\int\int_D \frac{1}{\sqrt{x^2+y^2}}dxdy$ becomes $\int_{\pi/3}^{\pi/2}\int_{4\cos\theta}^2 \frac{1}{r}rdrd\theta = \int_{\pi/3}^{\pi/2}\int_{4\cos\theta}^2 drd\theta = \int_{\pi/3}^{\pi/2}(2 - 4\cos\theta)d\theta = \frac{\pi}{3} - 4 + 2\sqrt{3}$.

2. Volume can be found as $\int\int_D(x^2 + y^2)dxdy$ where D is the interior of the disk. The bounds in polar coordinates are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 3$. The function $x^2 + y^2$ is r^2. So $V = \int_0^{2\pi}\int_0^3 r^2rdrd\theta = 2\pi\frac{r^4}{4}|_0^3 = \frac{81\pi}{2}$.

3. Volume can be found as $\int\int_D\sqrt{x^2+y^2}dxdy$ where D is the interior of the disk determined by the cylinder. The bounds in polar coordinates are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. The function $\sqrt{x^2+y^2}$ is r. So $V = \int_0^{2\pi}\int_0^2 rrdrd\theta = 2\pi\frac{r^3}{3}|_0^2 = \frac{16\pi}{3}$.

4. The paraboloid $z = 2 - x^2 - y^2$ is the upper surface and the cone $z = \sqrt{x^2 + y^2}$ is lower. Thus, the volume can be found as

$$V = \int\int (2 - x^2 - y^2 - \sqrt{x^2 + y^2}) dxdy.$$

The paraboloid and the cone intersect in a circle. The projection of the circle in xy-plane determines the bounds of integration.

Use the polar coordinates. In polar coordinates the paraboloid $2 - x^2 - y^2$ becomes $2 - r^2$ and the cone $\sqrt{x^2 + y^2}$ becomes r. They intersect when $2 - r^2 = r \Rightarrow 0 = r^2 + r - 2 = (r-1)(r+2) \Rightarrow r = 1$ (the negative solution -2 is not relevant). Thus, the bounds of integration are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$. The volume is $V = \int\int (2 - x^2 - y^2 - \sqrt{x^2 + y^2}) dxdy = \int_0^{2\pi} \int_0^1 (2 - r^2 - r)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 (2r - r^3 - r^2)\, dr = 2\pi\, (2\frac{r^2}{2} - \frac{r^4}{4} - \frac{r^3}{3})|_0^1 = 2\pi(1 - \frac{1}{4} - \frac{1}{3}) = 2\pi \frac{5}{12} = \frac{5\pi}{6}$.

5. The paraboloid $z = 36 - 3x^2 - 3y^2$ is the upper surface and the paraboloid $z = x^2 + y^2$ is the lower. Thus, $V = \int\int_D (36 - 3x^2 - 3y^2 - (x^2 + y^2)) dxdy$. The two surfaces intersect in a circle. The projection of the circle in xy-plane determines the bounds of integration. Use the polar coordinates. In polar coordinates $x^2 + y^2 = r^2$ and the surfaces become $z = 36 - 3r^2$ and $z = r^2$. They intersect when $36 - 3r^2 = r^2 \Rightarrow 36 = 4r^2 \Rightarrow 9 = r^2 \Rightarrow r = 3$ and $r = -3$ (negative solution is not relevant: r represents the distance so $r > 0$). Thus, the bounds of integration are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 3$. The volume is

$$V = \int\int (36 - 3r^2 - r^2) r dr d\theta = \int_0^{2\pi} d\theta \int_0^3 (36r - 4r^3)\, dr = 2\pi\, (18r^2 - r^4)\Big|_0^3 = 162\pi \approx 508.94.$$

6. The plane $z = 2 - y$ is the upper and the plane $z = 0$ is the lower surface. The cylinder $x^2 + y^2 = 1$ determines the region of the integration in xy-plane. Thus, $V = \int\int_D (2 - y - 0) dxdy = \int\int_D (2 - y) dxdy$ where D is the part of the unit disc in the first quadrant of the xy-plane. Using polar coordinates, this region is given by $0 \leq r \leq 1$, and $0 \leq \theta \leq \frac{\pi}{2}$. So, the volume is $V = \int\int_D (2 - y)\, dxdy = \int\int_D (2 - r\sin\theta)\, r dr d\theta = \int_0^{\pi/2} \int_0^1 (2r - r^2 \sin\theta) d\theta dr = \int_0^{\pi/2} (r^2 - \frac{r^3}{3} \sin\theta) d\theta \Big|_0^1 = \int_0^{\pi/2} (1 - \frac{1}{3}\sin\theta) d\theta = \theta + \frac{1}{3}\cos\theta \Big|_0^{\pi/2} = \frac{\pi}{2} - \frac{1}{3} \approx 1.24.$

7. Graph the curve first. From the graph, you can see that the bounds will be determined by the tangents to the curve. The tangents intersect the curve when $r = 0$. So, the bounds for θ can be obtained from the equation $\cos 2\theta = 0 \Rightarrow 2\theta = \pm\frac{\pi}{2} \Rightarrow \theta = \pm\frac{\pi}{4}$. So, the area is $A = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta d\theta = \frac{\pi}{8}$.