

Line Integrals with Respect to Arc Length

Suppose that C is a curve in xy -plane given by the equations $x = x(t)$ and $y = y(t)$ on the interval $a \leq t \leq b$. In this case think of $(x(a), y(a))$ as the **beginning** and $(x(b), y(b))$ as the **end** of C .

Recall that the length element ds is given by $ds = \sqrt{(x'(t))^2 + (y'(t))^2} dt$

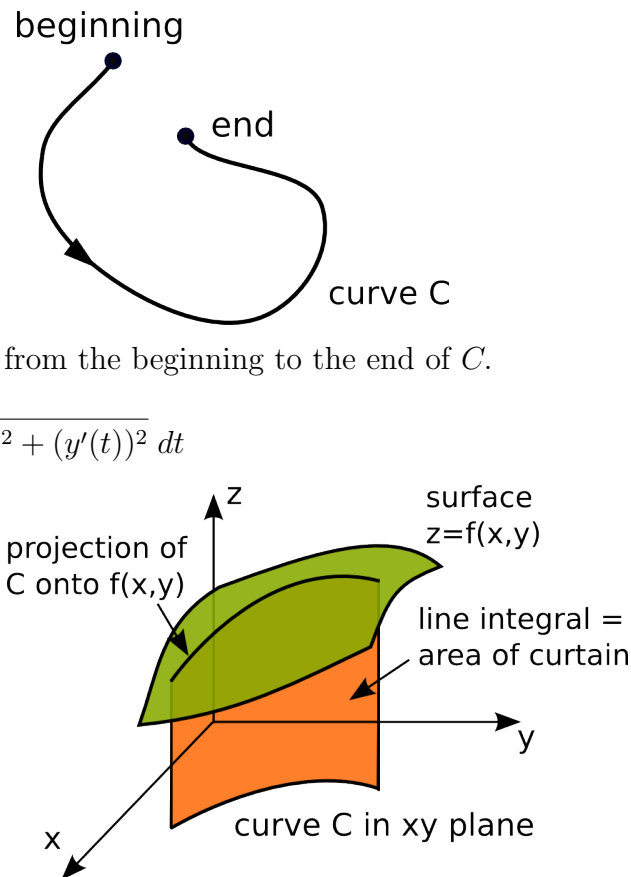
The total **length** of C can be found as integral of ds from the beginning to the end of C .

$$L(C) = \int_C ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Let $z = f(x, y)$ be a surface. The **line integral of C with respect to arc length** is

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

This integral represents the area of the surface between the curve C in the xy -plane and the projection of the curve C on the surface $z = f(x, y)$. This area is represented by the “curtain” in orange in the figure on the right.



Three dimensional curves. Suppose that C is a curve in space given by the equations $x = x(t)$, $y = y(t)$, and $z = z(t)$ on the interval $a \leq t \leq b$. Recall that the length of C is

$$L(C) = \int_C ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

Let $f(x, y, z)$ be a function. The **line integral of C with respect to arc length** is

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

Applications of line integrals. If a wire C in xy -plane has the density $\rho(x, y)$, then the mass m and the center of mass (\bar{x}, \bar{y}) are given by

$$m = \int_C \rho(x, y) ds \quad \bar{x} = \frac{1}{m} \int_C x \rho(x, y) ds \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y) ds.$$

If a wire C in space has the density $\rho(x, y, z)$, then the mass m and the center of mass $(\bar{x}, \bar{y}, \bar{z})$ are given by

$$m = \int_C \rho(x, y, z) ds \quad \bar{x} = \frac{1}{m} \int_C x \rho(x, y, z) ds \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y, z) ds \quad \bar{z} = \frac{1}{m} \int_C z \rho(x, y, z) ds.$$

Practice problems. Evaluate the following line integrals.

- $\int_C y ds$, $C : x = t^2, y = t, 0 \leq t \leq 2$
- $\int_C x y^4 ds$, C is the right half of the circle $x^2 + y^2 = 16$
- $\int_C x y^3 ds$, $C : x = 4 \sin t, y = 4 \cos t, z = 3t, 0 \leq t \leq \pi/2$
- $\int_C x e^{yz} ds$, C is the line segment from $(0,0,0)$ to $(1, 2, 3)$
- Find the mass and the center of mass of a wire in the shape of the curve C with the given density function ρ .
 - C is the right half of the circle $x^2 + y^2 = 4$ and the density function is a constant k .
 - C is the helix $x = 2 \sin t, y = 2 \cos t, z = 3t, 0 \leq t \leq 2\pi$, and the density function is a constant k .

Solutions.

- $x' = 2t, y' = 1$, so ds is $\sqrt{4t^2 + 1}$. $\int_C y ds = \int_0^2 t\sqrt{4t^2 + 1} dt = \frac{17^{3/2} - 1}{12} = 5.76$.
- The circle $x^2 + y^2 = 16$ parametrizes as $x = 4 \cos t, y = 4 \sin t$. So $ds = \sqrt{16 \sin^2 t + 16 \cos^2 t} = \sqrt{16} = 4$. Since we are integrating over its right half, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

$$\int_C x y^4 ds = \int_{-\pi/2}^{\pi/2} 4 \cos t 4^4 \sin^4 t 4 dt = 4^6 \frac{\sin^5 t}{5} \Big|_{-\pi/2}^{\pi/2} = \frac{8192}{5} = 1638.4.$$
- $x' = 4 \cos t, y' = -4 \sin t, z' = 3 \Rightarrow ds = \sqrt{16 \cos^2 t + 16 \sin^2 t + 9} = \sqrt{25} = 5$. $\int_C x y^3 ds = \int_0^{\pi/2} 4 \sin t 4^3 \cos^3 t 5 dt = (5)4^4 \frac{-\cos^4 t}{4} \Big|_0^{\pi/2} = (5)4^3 = 320$.
- The curve C is a line passing $(0, 0, 0)$ in the direction of the vector $\overrightarrow{PQ} = (1, 2, 3) - (0, 0, 0) = (1, 2, 3)$. So, an equation of this line is $x = 0 + 1t = t, y = 0 + 2t = 2t$ and $z = 0 + 3t = 3t$. So, $x' = 1, y' = 2, z' = 3$, and $ds = \sqrt{1 + 4 + 9} = \sqrt{14}$. The $t = 0$ is the t -value that corresponds to $(0,0,0)$ and $t = 1$ is the value that corresponds to $(1,2,3)$. So, the bounds are $0 \leq t \leq 1$. The integral is $\int_C x e^{yz} ds = \int_0^1 t e^{2t \cdot 3t} \sqrt{14} dt = \sqrt{14} \int_0^1 t e^{6t^2} dt = \sqrt{14} \frac{1}{12} e^{6t^2} \Big|_0^1 = \sqrt{14} \frac{e^6 - 1}{12} = 125.48$.
- a) The right half of the circle $x^2 + y^2 = 4$ parametrizes as $x = 2 \cos t, y = 2 \sin t$ with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. So $ds = \sqrt{4 \sin^2 t + 4 \cos^2 t} = \sqrt{4} = 2$. The mass is $m = \int_C k ds = \int_{-\pi/2}^{\pi/2} 2k dt = 2k\pi$. The x -coordinate is $\bar{x} = \frac{1}{2k\pi} \int_C k x ds = \frac{1}{2k\pi} \int_{-\pi/2}^{\pi/2} 2k 2 \cos t dt = \frac{8k}{2k\pi} = \frac{4}{\pi}$. The y -coordinate is $\bar{y} = \frac{1}{2k\pi} \int_C k y ds = \frac{1}{2k\pi} \int_{-\pi/2}^{\pi/2} 2k 2 \sin t dt = 0$.

b) $ds = \sqrt{4 \cos^2 t + 4 \sin^2 t + 9} = \sqrt{4 + 9} = \sqrt{13}$. The mass is $m = \int_C k ds = \int_0^{2\pi} k \sqrt{13} dt = 2k\pi \sqrt{13}$. The x -coordinate is $\bar{x} = \frac{1}{2k\pi \sqrt{13}} \int_C k x ds = \frac{1}{2k\pi \sqrt{13}} \int_0^{2\pi} k 2 \cos t \sqrt{13} dt = 0$. Similarly $\bar{y} = 0$. The z -coordinate is $\bar{z} = \frac{1}{2k\pi \sqrt{13}} \int_C k z ds = \frac{1}{2k\pi \sqrt{13}} \int_0^{2\pi} k 3t \sqrt{13} dt = \frac{6k\pi^2 \sqrt{13}}{2k\pi \sqrt{13}} = 3\pi$.