Calculus 3
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Line Integrals with Respect to Coordinates – Line Integrals of Vector Fields

Suppose that C is a curve in xy-plane given by the equations $x = x(t)$ and $y = y(t)$ on the interval $a \leq t \leq b$. The line integral over C of $z = f(x, y)$ with respect to x and y are is

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt \quad \text{and} \quad \int_C f(x,y)dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

These integrals are relevant when integrating *vector fields over curves.*

Vector fields. A (plane) vector field $\overrightarrow{f}$ is a function that assigns to each point (x, y) a two dimensional vector

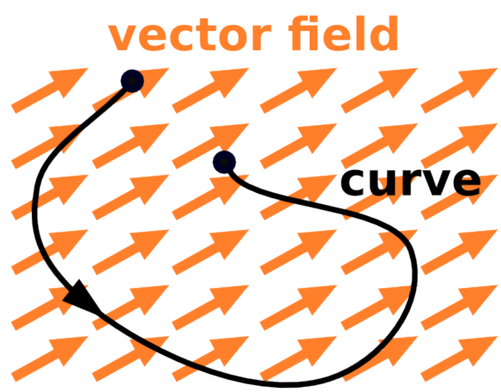
$$\overrightarrow{f}(x,y) = P(x,y)\,\overrightarrow{i} + Q(x,y)\,\overrightarrow{j} = \langle P(x,y), Q(x,y)\rangle$$

Example. An example of a vector field is the gradient vector of a two variable function. Recall that the gradient of $f(x, y)$. is $\nabla f(x,y) = f_x\,\overrightarrow{i} + f_y\,\overrightarrow{j} = \langle f_x, f_y\rangle$

Line integrals of vector fields. If we denote the position vector $\langle x, y\rangle$ by $\vec{r}$, the parametrization $x = x(t)$, $y = y(t)$ of a curve C can be represented simply as $\vec{r} = \langle x(t), y(t)\rangle$. The line integral $\int_C P dx + \int_C Q dy$ can be considered to be the integral of the dot product of the vector function $\langle P, Q\rangle$ and $d\vec{r} = \langle dx, dy\rangle$.

$$\int_C P dx + \int_C Q dy = \int_C (P dx + Q dy) =$$

$$\int_C \langle P, Q\rangle \cdot \langle dx, dy\rangle = \int_C \vec{f} \cdot d\vec{r}$$



This type of integrals measures **the total effect of a given field along a given curve.** In particular, many basic (non-continuous, one dimensional) formulas in physics can be represented in terms of line integrals in continuous and multi-dimensional cases. For example, $s = vt$ and its continuous version $s = \int v dt$ have more general version as $s = \int_C \vec{v} \cdot d\vec{r}(t)$ where $\vec{r}(t) = (x(t), y(t))$ is parametrization of the trajectory C.

Another example includes the formula computing the work done by the force $\overrightarrow{F}$ (possibly an electric or gravitational field) in moving the particle along the curve C

$$W = \int_C \overrightarrow{F} \cdot d\vec{r}.$$

Space vector fields. There is a three-dimensional version of all the concepts just discussed. First, a space vector field $\overrightarrow{f}$ is a function that assigns to each point (x, y, z) a three dimensional vector

$$\overrightarrow{f}(x,y,z) = P(x,y,z)\,\overrightarrow{i} + Q(x,y,z)\,\overrightarrow{j} + R(x,y,z)\,\overrightarrow{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

Example. An example of a space vector field is the gradient vector of a function $f(x,y,z)$. The gradient is $\nabla f(x,y,z) = f_x \overrightarrow{i} + f_y \overrightarrow{j} + f_z \overrightarrow{k} = \langle f_x, f_y, f_z \rangle$

Line integrals of space vector fields. If $\overrightarrow{f} = \langle P, Q, R \rangle$ is a vector field, and C a curve with parametrization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the line integral of $\overrightarrow{f}$ along C is

$$\int_C \overrightarrow{f} \cdot d\overrightarrow{r} = \int_C P\,dx + Q\,dy + R\,dz$$

To evaluate this integral, substitute $x(t), y(t)$, and $z(t)$ for every appearance of x, y, z. Recall that

$$dx = x'(t)dt,\ dy = y'(t)dt,\ \text{and } dz = z'(t)dt.\ \text{ So } d\vec{r} = \langle x'(t), y'(t), z'(t) \rangle dt.$$

The bounds of integration correspond to t-values which correspond to the beginning and the end of the curve C. You can find these in exactly the same way as finding the bounds for integral computing the length of a curve or the bounds of a line integral with respect to arc length in the previous section. If $t = a$ and $t = b$ correspond to the lower and the upper bound respectively, the line integral reduces to a single integral

$$\int_C \overrightarrow{f} \cdot d\overrightarrow{r} = \int_a^b \vec{f}(\vec{r}(t)) \cdot \overrightarrow{r}'(t)\,dt$$

or, using representation with P, Q, R

$$\int_C P\,dx + Q\,dy + R\,dz = \int_a^b P\,x'(t)dt + Q\,y'(t)dt + R\,z'(t)dt.$$

As in the two-dimensional case, this integral computes **the total effect of a given field along a given curve.** For example, if $\vec{f}$ is the force field, this integral computes the **work done by the force $\overrightarrow{f}$ in moving the particle along the curve** C.

$$W = \int_C \overrightarrow{f} \cdot d\overrightarrow{r}.$$

Line integrals of (scalar) functions versus vector fields. Let us compare the two types of line integrals. In the previous section, we have consider integrals in which integrand is a function which produces a value (scalar). In this section, the integrand is a function which produces a vector i.e. a vector field. Also, while the first type of integrals is independent of the orientation of the curve, the second type **depends on the orientation**. In particular, an integral of a vector function can be reduced to a single integral with larger parameter value as a lower and smaller value as an upper bound.

A scalar function is integrated with respect to the length element ds while a vector field is integrated with respect to the element $d\vec{r}$. These two elements are related by

$$ds = |\overrightarrow{r}'(t)|dt \quad \text{and} \quad d\vec{r} = \overrightarrow{r}'(t)dt.$$

Line integral of a scalar function $f(x,y,z)$	$\int_C f \ \ ds = \int_C f(\vec{r}(t))\,|\overrightarrow{r}'(t)|\ dt$
Line integral of a vector function $\vec{f}(x,y,z)$	$\int_C \vec{f} \cdot d\vec{r} = \int_C \vec{f}(\vec{r}(t)) \cdot \overrightarrow{r}'(t)\ dt$

The process of finding the parametrization and the bounds for each type of integrals is *exactly the same.* Just note that the order of bounds matters for the integral of a vector function.

Practice problems.

1. Evaluate the line integral where C is the given curve.

   a) $\int_C (xy + \ln x)\, dy$, $\quad C$ is the parabola $y = x^2$ from (1,1) to (3,9)

   b) $\int_C xy\, dx + (x - y)\, dy$, $\quad C$ consists of line segments from (0,0) to (2,0) and from (2,0) to (3,2)

   c) $\int_C x^3\, y^2 z\, dz$, $\quad C : x = 2t,\ y = t^2,\ z = t^2,\ 0 \leq t \leq 1$

   d) $\int_C z^2\, dx - z\, dy + 2y\, dz$, $\quad C$ consists of line segments from (0,0,0) to (0,1,1), from (0,1,1) to (1,2,3) and from (1,2,3) to (1, 2,4)

   e) $\int_C z^2\, dx + y\, dy + 2y\, dz$, $\quad$ where C consists of two parts C_1 and C_2. C_1 is the intersection of the cylinder $x^2 + y^2 = 16$ and the plane $z = 3$ from $(0, 4, 3)$ to $(-4, 0, 3)$. C_2 is a line segment from $(-4, 0, 3)$ to $(0, 1, 5)$.

2. Find the work done by the force field $\overrightarrow{f}$ in moving an object along the curve C.

   a) $\overrightarrow{f}(x, y) = (x - y)\ \overrightarrow{i} + xy\ \overrightarrow{j}$, $\quad C$ is the arc of the circle $x^2 + y^2 = 4$ traversed counter-clockwise from (2,0) to (0,2).

   b) $\overrightarrow{f}(x, y, z) = xz\ \overrightarrow{i} + xy\ \overrightarrow{j} + zy\ \overrightarrow{k}$, $\quad C : x = t^2,\ y = -t^3,\ z = t^4,\ 0 \leq t \leq 1$.

   c) $\overrightarrow{f}(x, y, z) = (x + y^2, y + z^2, z + x^2)$ and the curve C is the intersection of the plane $x + y + z = 1$ and the coordinate planes.

   d) $\overrightarrow{f} = (-y^2, x, z^2)$ and the curve C is the intersection of the plane $y+z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solutions.

1. a) With given parametrization $x = x$ and $y = x^2$, $dy = 2xdx$ and the bounds for x are $1 \leq x \leq 3$. So $\int_C (xy + \ln x)\ dy = \int_1^3 (x(x^2) + \ln x) 2xdx = \int_1^3 (2x^4 + 2x \ln x)dx = (\frac{2x^5}{5} + x^2 \ln x - \frac{x^2}{2})|_1^3 = (\frac{484}{5} + 9 \ln 3 - 4) = 102.69$

   b) The integral needs to be evaluated as a sum of two integrals since the two line segments have different parametrization. The line segment from (0,0) to (2,0) has an equation $x = x$, $y = 0$ and $0 \leq x \leq 2$. So, on this segment $dx = dx$ and $dy = 0$. The line segment from (2,0) to (3, 2) has an equation $x = x$, $y = 2x - 4$ and $2 \leq x \leq 3$. So, on this segment $dx = dx$ and $dy = 2dx$.

   $\int_C xy\ dx + (x - y)\ dy = \int_{C_1} xy\ dx + (x - y)\ dy + \int_{C_2} xy\ dx + (x - y)\ dy = \int_0^2 x0\ dx + (x - 0)\ 0 + \int_2^3 x(2x - 4)\ dx + (x - 2x + 4)\ 2dx = 0 + (\frac{2x^3}{3} - 3x^2 + 8x)|_2^3 = \frac{38}{3} - 7 = \frac{17}{3}$

   c) With given parametrization $x = 2t$, $y = t^2$, $z = t^2$, $dz = 2tdt$. So, the integral is $\int_C x^3\ y^2 z\ dz = \int_0^1 (2t)^3 (t^2)^2 (t^2)\, 2tdt = 16 \int_0^1 t^{10} dt = \frac{16}{11}$.

   d) The integral needs to be evaluated as a sum of three integrals since the three line segments have different parametrization. Let us denote the line segments by C_1, C_2 and C_3.

   The line segment C_1 is passing (0,0,0) in the direction of the vector $\overrightarrow{PQ} = (0, 1, 1) - (0, 0, 0) = (0, 1, 1)$. So C_1 has equations $x = 0$, $y = t$ and $z = t$ for $0 \leq t \leq 1$. So, on this segment $dx = 0$, $dy = dt$ and $dz = dt$. $\int_{C_1} z^2\ dx - z\ dy + 2y\ dz = \int_0^1 t^2(0) - tdt + 2tdt = \frac{1}{2}$.

The line segment C_2 is passing (0,1,1) in the direction of the vector $\overrightarrow{PQ} = (1,2,3)-(0,1,1) = (1,1,2)$. So C_2 has equations $x = t$, $y = 1+t$ and $z = 1+2t$ for $0 \le t \le 1$. So, on this segment $dx = dt$, $dy = dt$ and $dz = 2dt$. $\int_{C_2} z^2\, dx - z\, dy + 2y\, dz = \int_0^1 (1+2t)^2 dt - (1+2t)dt + 2(1+t)2dt = \int_0^1 (1+4t+4t^2-1-2t+4+4t)dt = \int_0^1 (6t+4t^2+4)dt = 3+\frac{4}{3}+4 = \frac{25}{3}$.

The line segment C_3 is passing (1,2,3) in the direction of the vector $\overrightarrow{PQ} = (1,2,4)-(1,2,3) = (0,0,1)$. So C_3 has equations $x = 1$, $y = 2$ and $z = 3+t$ for $0 \le t \le 1$. So, on this segment $dx = 0$, $dy = 0$ and $dz = dt$. $\int_{C_3} z^2\, dx - z\, dy + 2y\, dz = \int_0^1 (3+t)^2(0) - (3+t)(0) + 2(2)dt = 4$.

The final answer is $\frac{1}{2} + \frac{25}{3} + 4 = \frac{77}{6}$.

e) C_1 is on $x^2 + y^2 = 16$ thus $x = 4\cos t$ and $y = 4\sin t$. C_1 is also on $z = 3$ so $x = 4\cos t$, $y = 4\sin t$, $z = 3$ are parametric equations of C_1. On C_1, $dx = -4\sin t dt$, $dy = 4\cos t dt$ and $dz = 0$. The point (0,4,3) corresponds to $t = \frac{\pi}{2}$ and the point $(-4,0,3)$ to $t = \pi$. Thus, $\int_{C_1} z^2\, dx + y\, dy + 2y\, dz = \int_{\pi/2}^{\pi} 3^2(-4\sin t)dt + 4\sin t 4\cos t dt + 8\sin t(0) = (36\cos t + 8\sin^2 t)|_{\pi/2}^{\pi} = -36 - 8 = -44$.

The line segment C_2 is passing (-4,0,3) in the direction of the vector $\overrightarrow{PQ} = (0,1,5)-(-4,0,3) = (4,1,2)$. So C_2 has equations $x = -4+4t$, $y = t$ and $z = 3+2t$ for $0 \le t \le 1$. So, on this segment $dx = 4dt$, $dy = dt$ and $dz = 2dt$. $\int_C z^2\, dx + y\, dy + 2y\, dz = \int_0^1 (3+2t)^2 4dt + tdt + 2t2dt = \int_0^1 (36+53t+16t^2)dt = 36 + \frac{53}{2} + \frac{16}{3} = \frac{407}{6} = 67.83$.

So, the final answer is $\int_C = 67.83 - 44 = 23.83$.

2. a) C has a parametrization $x = 2\cos t$, $y = 2\sin t$. When $(x,y) = (2,0)$, $t = 0$ and when $(x,y) = (0,2)$ $t = \frac{\pi}{2}$. The work can be computed as $W = \int_C \vec{F} \cdot d\vec{r} = \int_C (x-y)dx + xydy = \int_0^{\pi/2} (2\cos t - 2\sin t)(-2\sin t)dt + 2\cos t 2\sin t 2\cos t dt = \int_0^{\pi/2} (-4\cos t\sin t + 4\sin^2 t + 8\cos^2 t \sin t)dt = (-2\sin^2 t + 2t - \sin 2t - \frac{8\cos^3 t}{3})|_0^{\pi/2} = -2 + \pi + \frac{8}{3} = 3.81$.

b) With the given parametrization $x = t^2$, $y = -t^3$, $z = t^4$, $0 \le t \le 1$, $dx = 2tdt$, $y = -3t^2 dt$, $z = 4t^3 dt$. The work can be computed as $W = \int_C \vec{F} \cdot d\vec{r} = \int_C xzdx + xydy + zydz = \int_0^1 t^2 t^4 2tdt + t^2 t^3 3t^2 - t^4 t^3 4t^3 dt = \int_0^1 (2t^7 + 3t^7 - 4t^{10})dt = (\frac{5t^8}{8} - \frac{4t^{11}}{11})_0^1 = \frac{5}{8} - \frac{4}{11} = \frac{23}{88}$.

c) The curve C consists of three parts C_1, C_2 and C_3 which are in the intersection of the plane and (1) xy-plane $z = 0$, (2) yz-plane $x = 0$, and (3) xz-plane $y = 0$, respectively. Positive orientation of C implies that C_1 is traversed from $(1,0,0)$ to $(0,1,0)$, C_2 from $(0,1,0)$ to $(0,0,1)$ and C_3 from $(0,0,1)$ to $(1,0,0)$.

On $C_1 : z = 0 \Rightarrow x+y+0 = 1 \Rightarrow\ y = 1-x$. Thus the line can be parametrized by $x = x, y = 1-x, z = 0$. So, $dx = dx$, $dy = -dx$ and $dz = 0$ and the bounds are from 1 to 0. Hence, $\int_{C_1} = \int_{C_1} (x+y^2)dx + (y+z^2)dy + (z+x^2)dz = \int_1^0 (x+(1-x)^2)dx + (1-x)(-1)dx = \int_1^0 (x+1-2x+x^2-1+x)dx = \int_1^0 x^2 dx = \frac{-1}{3}$.

On $C_2 : x = 0 \Rightarrow 0+y+z = 1 \Rightarrow\ z = 1-y$. Thus the line can be parametrized by $x = 0, y = y, z = 1-y$ $\Rightarrow dx = 0$, $dy = dy$ and $dz = -dy$. The bounds are from 1 to 0. So, $\int_{C_2} = \int_{C_2} (x+y^2)dx + (y+z^2)dy + (z+x^2)dz = \int_1^0 (y+(1-y)^2)dy + (1-y)(-1)dy = \int_1^0 (y+1-2y+y^2-1+y) = \int_1^0 y^2 dy = \frac{-1}{3}$.

On $C_3 : y = 0 \Rightarrow x+0+z = 1 \Rightarrow\ z = 1-x$. Thus the line can be parametrized by $x = x, y = 0, z = 1-x$ $\Rightarrow dx = dx$, $dy = 0$ and $dz = -dx$. The bounds are from 0 to 1. So, $\int_{C_3} = \int_{C_3} (x+y^2)dx + (y+z^2)dy + (z+x^2)dz = \int_0^1 xdx + (1-x+x^2)(-1)dx = \int_0^1 (2x-1-x^2)dx = 1 - 1 - \frac{1}{3} = \frac{-1}{3}$.

Thus $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{-1}{3} - \frac{1}{3} - \frac{1}{3} = -1$.

d) C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - y = 2 - \sin t$, $0 \le t \le 2\pi$. $\int_C = \int_C -y^2 dx + xdy + z^2 dz = \int_0^{2\pi} \sin^3 tdt + \cos^2 tdt + (2-\sin t)^2 \cos tdt = \pi$.