Calculus 3
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Power Series

A **power series** is a series of the form $\sum_{n=1}^{\infty} a_n x^n$.

More generally, a series of the form $\sum_{n=1}^{\infty} a_n (x-a)^n$ is called a **power series centered at** a.

Note that this is **a function of** x. This function is **defined** for all values of x for which the series **converges** and, for a given power series, exactly one of the following three cases holds:

1. There is a positive number R such that the series converges if $|x-a| < R$ (that is $-R < x-a < R \Rightarrow a-R < x < a+R$) and diverges if $|x-a| > R$. Such number R is called the **radius of convergence**. Note that the **endpoints** of the interval $(a-R, a+R)$ have to be checked separately. You can use the appropriate tests we covered earlier.

2. The series converges for all values of x (that is on the interval $(-\infty, \infty)$). In this case, R can be considered to be ∞.

3. The series converges just when $x = a$. In this case, R can be considered to be 0.

Determining the interval of convergence boils down to determining the radius of convergence. In many cases, the use of the Ratio or the Root Tests is a good way to start finding the radius R. Keep in mind that those tests require the series to have positive terms, so start by considering $\sum_{n=1}^{\infty} |a_n x^n|$.

Practice Problems. Find the intervals where the following power series converge.

1. $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$
2. $\sum_{n=1}^{\infty} \frac{x^n}{n}$
3. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$
4. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}$
5. $\sum_{n=1}^{\infty} \frac{(x-2)^{n+1}}{n3^n}$
6. $\sum_{n=1}^{\infty} \frac{3^n x^n}{n+1}$
7. $\sum_{n=1}^{\infty} (-1)^n \, n \, 2^n x^n$

Solutions.

1. The series can be considered as a geometric series with $r = \frac{x}{2}$. Thus, it is convergent for $-1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$ and divergent for $x \geq 2$ and $x \leq -2$.

   Alternatively, you can use the root test for $\sum_{n=1}^{\infty} |\frac{x}{2}|^n$ and evaluate $\lim_{n\to\infty} \sqrt[n]{|\frac{x}{2}|^n} = |\frac{x}{2}|$. This series is convergent for $|\frac{x}{2}| < 1 \Rightarrow -1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$ and divergent for $|\frac{x}{2}| > 1 \Rightarrow x \geq 2$ and $x \leq -2$. Then you have to check the endpoints $|\frac{x}{2}| = 1 \Rightarrow x = \pm 2$ since the root test is inconclusive in this case. When $x = 2$, the series becomes $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = 1 + 1 + \ldots$ which is divergent either by the divergence test or by the geometric series test ($r = 1$). When $x = -2$, the series becomes $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n$ which is divergent either by the divergence test or by the geometric series test ($r = -1$).

2. Use the Ratio Test for $\sum_{n=1}^{\infty} |\frac{x^n}{n}| = \sum_{n=1}^{\infty} \frac{|x|^n}{n}$ to determine the interval of convergence $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)} \frac{n}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{(n+1)} \frac{n}{1} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|\frac{1}{1} = |x|$. Thus, the series is convergent when $|x| < 1 \Rightarrow -1 < x < 1$. The series is divergent if $x > 1$ or $x < -1$. Check the convergence on the endpoints of the interval (-1,1).

   When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is a p-series with $p = 1$. Since $p = 1 \leq 1$, the series is divergent. When $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is an alternating series with $b_n = \frac{1}{n}$. Since $\frac{1}{n}$ is decreasing and converges to 0, the series is convergent by the Alternating Series Test. Thus, the interval of convergence is $-1 \leq x < 1$.

3. Use the Ratio Test for $\sum_{n=1}^{\infty} |\frac{x^n}{n!}| = \sum_{n=1}^{\infty} \frac{|x|^n}{n!}$ to determine the interval of convergence: $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{1\cdot 2\cdot \ldots \cdot n\cdot (n+1)} \frac{1\cdot 2\cdot \ldots \cdot n}{1} = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x|\frac{1}{\infty} = 0$. Since $0 < 1$, this is convergent for any value of x (i.e. the interval of convergence is $(-\infty, \infty)$). So, this is an example of a series with infinite radius of convergence.

4. Use the Ratio Test for $\sum_{n=1}^{\infty} \frac{|x-1|^n}{n2^n}$. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)\ 2^{n+1}} \frac{n\ 2^n}{|x-1|^n} = \lim_{n\to\infty} \frac{|x-1|}{(n+1)2} \frac{n}{1} = |x-1| \lim_{n\to\infty} \frac{n}{2n+2} = |x-1|\frac{1}{2} = \frac{|x-1|}{2}$. Thus, the series is convergent when $\frac{|x-1|}{2} < 1 \Rightarrow |x-1| < 2 \Rightarrow -2 < x-1 < 2 \Rightarrow -1 < x < 3$. The series is divergent if $x > 3$ or $x < -1$. Check the convergence on the endpoints of the interval (-1,3).

   When $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is a p-series with $p = 1$. Since $p = 1 \leq 1$, the series is divergent. When $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is an alternating series with $b_n = \frac{1}{n}$. Since $\frac{1}{n}$ is decreasing and converges to 0, the series is convergent by the Alternating Series Test. Thus, the interval of convergence is $-1 \leq x < 3$.

5. Use the Ratio Test for $\sum_{n=1}^{\infty} \frac{|x-2|^{n+1}}{n3^n}$. $\lim_{n\to\infty} \frac{|x-2|^{n+2}}{(n+1)\ 3^{n+1}} \frac{n\ 3^n}{|x-2|^{n+1}} = \lim_{n\to\infty} \frac{|x-2|}{(n+1)3} \frac{n}{1} = |x-2| \lim_{n\to\infty} \frac{n}{3n+3} = \frac{|x-2|}{3}$. Thus, the series is convergent when $\frac{|x-2|}{3} < 1 \Rightarrow |x-2| < 3 \Rightarrow -3 < x-2 < 3 \Rightarrow -1 < x < 5$. The series is divergent if $x > 5$ or $x < -1$. Checking the convergence on the endpoints is similar to the previous problem. For $x = 5$, using the p-test you can obtain that the series is divergent and for $x = -1$, you can use the Alternating Series Test to show convergence. Thus, the interval of convergence is $-1 \leq x < 5$.

6. The problem is similar to previous two. Using the Ratio Test obtain that the series is convergent when $3|x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow \frac{-1}{3} < x < \frac{1}{3}$. Test the endpoints then. When $x = \frac{1}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n+1}$ which can be tested using the Integral Test or by noting that $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$ and using the p-test. In either case, the series is divergent for $x = \frac{1}{3}$. When $x = \frac{-1}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$. Since $\frac{1}{n+1}$ is decreasing and converges to 0, the series is convergent by the Alternating Series Test. Thus, the interval of convergence is $\frac{-1}{3} \leq x < \frac{1}{3}$.

7. Use the Ratio Test for $\sum_{n=1}^{\infty} |-1|^n n2^n |x|^n = \sum_{n=1}^{\infty} n2^n|x|^n$. $\lim_{n\to\infty} \frac{(n+1)2^{n+1}|x|^{n+1}}{n2^n|x|^n} = \lim_{n\to\infty} \frac{(n+1)2|x|}{n} = 2|x| \lim_{n\to\infty} \frac{n+1}{n} = 2|x|$. Thus, the series is convergent when $2|x| < 1 \Rightarrow |x| < \frac{1}{2} \Rightarrow \frac{-1}{2} < x < \frac{1}{2}$. Test the endpoints then. When $x = \frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n n$ which can be tested using the Divergence test. Since the limit of the n-th term does not exist, the series is divergent. When $x = \frac{-1}{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n n(-1)^n = \sum_{n=1}^{\infty} n$ which can be tested using the Divergence test. Since the limit of the n-th term is infinity, the series is divergent. Thus, the interval of convergence is $\frac{-1}{2} < x < \frac{1}{2}$.