Calculus 3
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Review for Exam 1

1. **Surfaces.** Describe the following surfaces.
   (a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 + z^2 = 9$ (c) $z = 3$
   (d) $z = 6 - 3x - 2y$ (e) $z = \sqrt{16 - x^2 - y^2}$ (f) $z = x^2 + y^2$
   (g) $z = \sqrt{x^2 + y^2}$

2. **Review of Vectors.**
   (a) Let $\overrightarrow{a} = \langle 3, 4, 0\rangle$ and $\overrightarrow{b} = \langle -1, 4, 2\rangle$. Find $|\overrightarrow{a}|$, $2\overrightarrow{a} + 3\overrightarrow{b}$, $3\overrightarrow{a} - 2\overrightarrow{b}$. Find the normalization of $\overrightarrow{a}$.
   (b) Find $\overrightarrow{a} \cdot \overrightarrow{b}$ and $\overrightarrow{a} \times \overrightarrow{b}$ for the two vectors from previous problem. Determine if the vectors are parallel, orthogonal or neither.
   (c) Let $\overrightarrow{a} = \langle 1, 0, 1\rangle$ and $\overrightarrow{b} = \langle 1, -1, 0\rangle$. Find $\overrightarrow{a} \cdot \overrightarrow{b}$. Find the projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$.

3. **Lines and Planes.**
   (a) Find an equation of the line through the point $(-2, 4, 10)$ and parallel to the vector $\langle 3, 1, -8\rangle$.
   (b) Find an equation of the line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.
   (c) Find an equation of the line through the points $(3, 1, -1)$ and $(3, 2, -6)$.
   (d) Find an equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5\rangle$.
   (e) Find an equation of the plane through the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$.
   (f) Find an equation of the plane through the points $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.
   (g) Find an equation of the line of the intersection of the planes $x+y-z = 0$ and $2x-5y-z = 1$.

4. **Curves in Space.**
   (a) Consider the curve $x = \cos t \quad y = \sin t \quad z = t$. Find an equation of the tangent line to the curve at the point where $t = 0$. Find the length of the curve from $t = 0$ to $t = 1$.
   (b) Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$. Find parametric equations of this curve and an equation of the tangent line to the curve at the point where $t = 0$ in parametrization. Using the calculator, estimate the length of the curve from $t = 0$ to $t = \pi/2$.

(c) Consider the curve C which is the intersection of the surfaces $x^2+y^2=9$ and $z=1-y^2$. i) Find the parametric equations that represent the curve C. ii) Find the equation of the tangent line to the curve C at point $(0,3,-8)$. iii) Find the length of the curve from $(3,0,1)$ to $(0,3,-8)$. You can use the calculator for the integral that you are going to get.

(d) Consider the curve C which is the intersection of the surfaces

$$y^2+z^2=16 \qquad \text{and} \qquad x=8-y^2-z$$

i) Find the parametric equations that represent the curve C. ii) Find the equation of the tangent line to the curve C at point $(-8,-4,0)$. iii) Find the length of the curve from $(4,0,4)$ to $(-8,-4,0)$. Use the calculator to evaluate the integral that you are going to get.

(e) Find the length of the boundary of the part of the paraboloid $z=4-x^2-y^2$ in the first octant.

5. **Partial Derivatives.** Find the indicated derivatives.

   (a) $z=3x^2+2xy-5y^2$; z_x, z_y, z_{xx}, z_{xy}, z_{yx} and z_{yy}.

   (b) $z=e^x\sin y$; z_x, z_y, z_{xx}, z_{xy}, z_{yx} and z_{yy}.

   (c) $z=ax^2e^{x^2-xy}$ where a is a constant; z_x, z_y, z_{xx}, z_{xy} and z_{yy}.

   (d) $z=x\ln(xy^2)$; z_x, z_y, z_{xx}, z_{xy} and z_{yy}.

   (e) $xy^2+yz^2+zx^2=3$; z_x and z_y at $(1,1,1)$.

   (f) $x-yz=\cos(x+y+z)$; z_x and z_y at $(0,1,-1)$.

   (g) $z=3x^2+2xy-5y^2$, $x=2+t^2$, $y=1-t^3$; $z'(t)$ when $t=0$.

   (h) $z=x\ln(x+2y)$, $x=\cos t$, $y=\sin t$; $z'(t)$ when $t=0$.

   (i) $z=e^xy+xy^2$, $x=st$, $y=s^2+t^2$; z_s and z_t at $(1,1)$.

   (j) $z=x^2+xy$, $x=e^t\cos s$, $y=e^t\sin s$; z_s and z_t at $(\pi,0)$.

6. **Tangent planes.** Find the equation of the tangent plane to a given surface at a specified point.

   (a) $z=y^2-x^2$, at $(-4,5,9)$

   (b) $z=e^x\ln y$, at $(3,1,0)$

   (c) $x^2+2y^2+3z^2=21$, at $(4,-1,1)$

   (d) $xy^2+yz^2+zx^2=3$; at $(1,1,1)$.

   (e) $x-yz=\cos(x+y+z)$; at $(0,1,-1)$.

7. **Linear Approximation.**

   (a) If $f(2,3)=5$, $f_x(2,3)=4$ and $f_y(2,3)=3$, approximate $f(2.02,3.1)$.

   (b) If $f(1,2)=3$, $f_x(1,2)=1$ and $f_y(1,2)=-2$, approximate $f(.9,1.99)$.

   (c) Find the linear approximation of $z=\sqrt{20-x^2-7y^2}$ at $(2,1)$ and use it to approximate the value at $(1.95,1.08)$.

   (d) Find the linear approximation of $z=\ln(x-3y)$ at $(7,2)$ and use it to approximate the value at $(6.9,2.06)$.

8. **Applications.**

(a) The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is rising at a rate of 0.15 K/s. The pressure P, volume V and temperature T are related by the equation $PV = 8.31T$. Find the rate of change of the volume when the pressure is 20 kPa and temperature 320 K.

(b) The number N of bacteria in a culture depends on temperature T and pressure P which depend on time t in minutes. Assume that 3 minutes after the experiment started, the pressure is increasing at a rate of 0.1 kPa/min and the temperature at a rate of 0.5 K/min. The number of bacteria changes at the rates of 3 bacteria per kPa and 5 bacteria per Kelvin. i) Find the rate at which the number of bacteria is increasing 3 minutes after the experiment started.

ii) Assume that the rates of 3 bacteria per kPa and 5 bacteria per Kelvin are constant. If there is 300 bacteria initially when $T = 305$ K and $P = 102$ kPa, estimate the number of bacteria when $T = 309$ K and $P = 100$ kPa.

(c) The number of flowers N in a closed environment depends on the amount of sunlight S that the flowers receive and the temperature T of the environment. Assume that the number of flowers changes at the rates $N_S = 2$ and $N_T = 4$. i) If there are 100 flowers when $S = 50$ and $T = 70$, estimate the number of flowers when $S = 52$ and $T = 73$.

ii) If the temperature depends on time as $T(t) = 85 - \frac{8}{1+t^2}$ and the amount of sunlight decreases on time as $S(t) = \frac{1}{t}$, find the rate of change of the flower population at time $t = 2$ days.

(d) The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$ where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. Determine how fast the temperature rises on the bug's path after 3 seconds.

(e) An object moves in the space away from its initial position so that after t hours it is at $x = 2t$ and $y = 2t^2 - 1$ and $z = \sqrt{3t+1}$ miles from its initial position. Find the speed of that object 5 hours after it started moving. (Recall that the speed is the length of the tangent vector at a point.)

Solutions

More detailed solutions of the problems can be found on the class handouts.

1. Surfaces.
   (a) Cylinder, base is a circle $x^2 + y^2 = 4$ in xy-plane. (b) Sphere, center at origin, radius 3.
   (c) Horizontal plane, passes $(0, 0, 3)$. (d) Plane, passes $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 6)$.
   (e) Upper half-sphere, center is at origin, radius 4. (f) Paraboloid, obtained by rotating a parabola $z = y^2$ in zy-plane about z-axis.
   (g) Cone, obtained by rotating a line $z = y$ in zy-plane about z-axis.

2. Review of Vectors.
   (a) $|\overrightarrow{a}| = 5$, $2\overrightarrow{a} + 3\overrightarrow{b} = \langle 3, 20, 6\rangle$, $3\overrightarrow{a} - 2\overrightarrow{b} = \langle 11, 4, -4\rangle$. $\frac{\vec{a}}{|\vec{a}|} = \langle 3/5, 4/5, 0\rangle$.
   (b) $\overrightarrow{a} \cdot \overrightarrow{b} = 13$, $\overrightarrow{a} \times \overrightarrow{b} = \langle 8, -6, 16\rangle$. $\overrightarrow{a}$ and $\overrightarrow{b}$ are neither parallel nor orthogonal.
   (c) $\overrightarrow{a} \cdot \overrightarrow{b} = 1$ The projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$ is $\langle 1/2, 0, 1/2\rangle$.

3. Lines and Planes.
   (a) $x = -2 + 3t \quad y = 4 + t \quad z = 10 - 8t$ (b) $x = 1 + t \quad y = 3t \quad z = 6 + t$
   (c) $x = 3 \quad y = 1 + t \quad z = -1 - 5t$ (d) $-2x + y + 5z = 1$
   (e) $3x - 7z = -9$ (f) $x + y + z = 2$ (g) $x = 1 + 6t \quad y = t \quad z = 1 + 7t$

4. Curves in Space.
   (a) Tangent: $x = 1$, $y = t$, $z = t$. Length: $\sqrt{2}$ (b) Tangent: $x = 1$, $y = t$, $z = 2 - t$. Length: 1.91
   (c) i) $x = 3\cos t$, $y = 3\sin t$, $z = 1 - y^2 = 1 - 9\sin^2 t$. ii) $(0, 3, -8)$ corresponds to $t = \pi/2$. Plugging $\pi/2$ in derivative gives you $\langle -3, 0, 0\rangle$. Tangent line: $x = -3t$ $y = 3$ $z = -8$. iii) $(3, 0, 1)$ corresponds to $t = 0$ and $(0, 3, -8)$ to $t = \pi/2$. So, the bounds of integration are 0 to $\pi/2$. The length is 10.48.
   (d) i) $y = 4\cos t$, $z = 4\sin t$, $x = 8 - y^2 - z = 8 - 16\cos^2 t - 4\sin t$. ii) $(-8, -4, 0)$ corresponds to $t = \pi$. Plugging π in derivative gives you $\langle 4, 0, -4\rangle$. Tangent line: $x = 4t - 8$ $y = -4$ $z = -4t$. iii) $(4, 0, 4)$ corresponds to $t = \pi/2$ and $(-8, -4, 0)$ to $t = \pi$. So, the bounds of integration are $\pi/2$ to π. The length is 14.515.
   (e) A set of parametric equations for the three curves in the intersection is
   $x = 2\cos t, y = 2\sin t, z = 0$ with $0 \le t \le \frac{\pi}{2}$,
   $x = t, y = 0, z = 4 - t^2$ with $0 \le t \le 2$, and
   $x = 0, y = t, z = 4 - t^2$ with $0 \le t \le 2$.
   The three derivative vectors and length elements are
   $x' = -2\sin t, y' = 2\cos t, z' = 0 \Rightarrow \quad ds = \sqrt{4\sin^2 t + 4\cos^2 t}dt = \sqrt{4}dt = 2dt$
   $x' = 1, y' = 0, z' = -2t \Rightarrow \quad ds = \sqrt{1 + 4t^2}dt$, and
   $x' = 0, y' = 1, z' = -2t \Rightarrow \quad ds = \sqrt{1 + 4t^2}dt$.
   The total length is $\int_0^{\pi/2} 2dt + \int_0^2 \sqrt{1 + 4t^2}dt + \int_0^2 \sqrt{1 + 4t^2}dt = \pi + 4.65 + 4.65 \approx 12.44$.

5. Partial Derivatives.

(a) $z_x = 6x + 2y, \quad z_y = 2x - 10y, \quad z_{xx} = 6, \quad z_{xy} = z_{yx} = 2, \quad z_{yy} = -10$

(b) $z_x = e^x \sin y, \quad z_y = e^x \cos y, \quad z_{xx} = e^x \sin y, \quad z_{xy} = z_{yx} = e^x \cos y, \quad z_{yy} = -e^x \sin y$

(c) $z_x = 2axe^{x^2-xy} + ax^2e^{x^2-xy}(2x - y) = a(2x + 2x^3 - x^2y)e^{x^2-xy}$, $z_y = ax^2e^{x^2-xy}(-x) = -ax^3e^{x^2-xy}$. Then $z_{xx} = a(2+6x^2-2xy)e^{x^2-xy}+a(2x+2x^3-x^2y)e^{x^2-xy}(2x-y)$ and $z_{yy} = -ax^3e^{x^2-xy}(-x) = ax^4e^{x^2-xy}$. Differentiating z_x with respect to y get $z_{xy} = a(-x^2)e^{x^2-xy}+a(2x+2x^3-x^2y)e^{x^2-xy}(-x) = a(-x^2-2x^2-2x^4+x^3y)e^{x^2-xy} = a(-3x^2-2x^4+x^3y)e^{x^2-xy}$. Alternatively, differentiating z_y with respect to x get $z_{xy} = -3ax^2e^{x^2-xy} - ax^3e^{x^2-xy}(2x-y) = a(-3x^2 - 2x^4 + x^3y)e^{x^2-xy}$

(d) $z_x = \ln(xy^2) + 1, \quad z_y = 2x/y, \quad z_{xx} = 1/x, \quad z_{xy} = z_{yx} = 2/y, \quad z_{yy} = -2x/y^2$

(e) $z_x = -(y^2+2xz)/(2yz+x^2)$, $z_y = -(2xy+z^2)/(2yz+x^2)$. At $(1,1,1)$, $z_x = -1$, $z_y = -1$.

(f) $z_x = -(1+\sin(x+y+z))/(-y+\sin(x+y+z))$ and $z_y = -(-z+\sin(x+y+z))/(-y+\sin(x+y+z))$. At $(0,1,-1)$, $z_x = 1$ and $z_y = 1$.

(g) $z'(t) = (6x + 2y)(2t) + (2x - 10y)(-3t^2); \quad z'(0) = 0$

(h) $z'(t) = (\ln(x + 2y) + x/(x + 2y))(-\sin t) + (2x/(x + 2y))(\cos t); \quad z'(0) = 2.$

(i) $z_s = (e^xy + y^2)t + (e^x + 2xy)2s$; $z_t = (e^xy + y^2)s + (e^x + 2xy)2t$; $z_s(1,1) = 4e + 12$ and $z_t(1,1) = 4e + 12$

(j) $z_s = (2x + y)(-e^t \sin s) + x(e^t \cos s)$, $z_t = (2x + y)(e^t \cos s) + x(e^t \sin s)$, $z_s(\pi, 0) = 1$ and $z_t(\pi, 0) = 2$

6. Tangent planes.

(a) $8x + 10y - z = 9$

(b) $e^3y - z = e^3$

(c) $F_x = 2x, F_y = 4y, F_z = 6z$. At $(4,-1,1)$ this produces vector $\langle 8,-4,6\rangle$. The tangent plane is $4x - 2y + 3z = 21$.

(d) $F_x = y^2+2xz, F_y = 2xy+z^2, F_z = 2yz+x^2$. At $(1,1,1)$ this produces vector $\langle 3,3,3\rangle$. The tangent plane $x + y + z = 3$.

(e) $F_x = 1+\sin(x+y+z), F_y = -z+\sin(x+y+z), F_z = -y+\sin(x+y+z)$. At $(0,1,-1)$ this produces vector $\langle 1,1,-1\rangle$. The tangent plane is $x + y - z = 2$.

7. Linear Approximation. (a) $f(2.02, 3.1) \approx 5.38$ (b) $f(.9, 1.99) \approx 2.92$ (c) $f(1.95, 1.08) \approx 2.847$ (d) $f(6.9, 2.06) \approx -0.28$

8. Applications.

(a) -.27 liter per second

(b) i) Since $P'(3) = 0.1$, $T'(3) = 0.5$, $N_P = 3$, and $N_T = 5$, and $N'(t) = N_TT'+N_PP'$ We have that $N'(3) = 5\cdot 0.5+3\cdot 0.1 = 2.8$ bacteria/minute. ii) Using linear approximation formula, $N(T,P) \approx N(T_0,P_0) + N_T \cdot (T - T_0) + N_P \cdot (P - P_0) \Rightarrow N(309,100) \approx N(305,102) + 5(309 - 305) + 3(100 - 102) = 300 + 5(4) + 3(-2) = 314$ bacteria.

(c) i) $100 + 2(52 - 50) + 4(73 - 70) = 116$ flowers. ii) $N' = N_SS' + N_TT' = 2\frac{-1}{4} + 4\frac{32}{25} = 4.62$ flowers/day.

(d) 2 degrees Celsius per second (e) Speed = 20.1 miles per hour.