Calculus 3
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Review for the Final Exam

1. **Sequences.** Determine whether the following sequences are convergent or divergent. If they are convergent, find their limits.

   (a) $a_n = \left(\frac{1}{2}\right)^n$

   (b) $a_n = \frac{n+1}{2n-1}$

   (c) $a_n = \frac{n+3n^3}{4n^2+35n-7+2n^3}$,

   (d) $a_1 = 1, \quad a_{n+1} = \frac{1}{1+a_n}$

   (e) $a_0 = 0, \quad a_{n+1} = \sqrt{2+a_n}$

   (f) When calculating the hydrogen ion concentration $[\mathrm{H}^+]$ in a acid-base system, the problem frequently boils down to finding the limit of a recursive sequence. For example, when hydrochloric acid HCl is dissolved in water, we have $[\mathrm{H}^+]_1 = C_{\mathrm{HCl}}$ and

   $$[\mathrm{H}^+]_{n+1} = C_{\mathrm{HCl}} + \frac{\mathrm{K}_w}{[\mathrm{H}^+]_n},$$

   where C_{HCl} is the analytical concentration of HCl and K_w is the water's autoprotolysis constant that is equal to 10^{-14} at 25 degrees Centigrade. If the analytical concentration of HCL C_{HCl} is equal to 10^{-7}, find the hydrogen ion concentration $[\mathrm{H}^+]$ and its pH value.

2. **Sum of Series.** Find the sum of the following series.

   (a) $\sum_{n=2}^{\infty} \frac{2^{n+2}}{3^{n-1}}$

   (b) $\sum_{n=1}^{\infty} 2^{2n}5^{-n}$

   (c) $2 - 2/3 + 2/9 - 2/27 + 2/81 - \ldots$

   (d) $3 - 3/4 + 3/16 - 3/64 + 3/256 - \ldots$

3. **Convergence of Series.** Determine whether the following series are convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \frac{13n^2}{n^2+4n+5}$

   (b) $\sum_{n=0}^{\infty} (1/2)^n$

   (c) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

   (d) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

   (e) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$

   (f) $\sum_{n=1}^{\infty} \frac{1}{(n+3)^4}$

   (g) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

   (h) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n+1}$

   (i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2+1}$

(j) $\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \dots$

(k) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(l) $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

4. **Convergence of Power Series.** Find all the values of x for which the series converges.

   (a) $\sum_{n=1}^{\infty} 4^n x^n$

   (b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}$

   (c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

   (d) $\sum_{n=1}^{\infty} \frac{3^n x^n}{n+1}$

   (e) $\sum_{n=1}^{\infty} \frac{(x-2)^{n+1}}{n3^n}$

5. **Power Series Expansion.** Find the power series expansion of the following functions centered at given point.

   (a) $e^{2x}; \quad x = 0$

   (b) $xe^{2x}; \quad x = 0$

   (c) $\frac{1}{1-x^2}; \quad x = 0$

   (d) $\frac{1}{1+x}; \quad x = 0$

   (e) $\sin 3x; \quad x = 0$

   (f) $\int_0^x e^{x^2}\, dx; \quad x = 0$

6. **Applications of Taylor Polynomials.**

   (a) Find the Taylor polynomial of the third degree centered at 0 for e^x. Using the polynomial, evaluate $e^{0.3}$.

   (b) Find the Taylor polynomial of the fourth degree centered at 0 for $\sin x$. Using the polynomial, evaluate $\sin(0.2)$.

   (c) Find the Taylor polynomial of the second degree centered at 0 for $e^x \sin x$. Using the polynomial, evaluate $e^{1/2} \sin(1/2)$.

   (d) If $f(2) = 5$, $f'(2) = 3$ and $f''(2) = 1$, approximate $f(2.1)$.

   (e) If $f(2) = 5$, $f'(2) = 3$, $f''(2) = 1$, and $f'''(x) = 1/2$ approximate $f(1.9)$.

   (f) If $f(1) = f'(1) = -1$, $f''(1) = f'''(1) = 0$ and $f^{iv}(1) = 2$, approximate $f(1.01)$.

   (g) Approximate the function $e^{\frac{hv}{kT}} - 1$ with the Taylor polynomial of the second degree in terms of v.

   (h) The magnitude of the electric field E of a single charge q can be described by $E = \frac{kq}{r^2}$ where r is the distance between the field and the charge and k is a proportionality constant. If two opposite charges on distance d from each other create an electric dipole moment, this formula changes to

   $$E = \frac{kq}{(r-d)^2} - \frac{kq}{(r+d)^2} = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2}$$

   Use the Taylor polynomial of the second degree of the function $f(x) = \frac{1}{(1-x)^2}$ to show that the magnitude of the electric field E can be approximated as $E \approx \frac{4kqd}{r^3}$.

7. **Lines and Planes.**

   (a) Find an equation of the line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.

   (b) Find an equation of the line through the points $(3, 1, -1)$ and $(3, 2, -6)$.

   (c) Find an equation of the plane through the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$.

   (d) Find an equation of the plane through the points $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.

8. **Curves in Space.**

   (a) Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$. Find an equation of the tangent line to the curve at the point where $t = 0$. Using the calculator, estimate the length of the curve from $t = 0$ to $t = \pi/2$.

   (b) Consider the curve C which is the intersection of the surfaces

   $$y^2 + z^2 = 16 \qquad \text{and} \qquad x = 8 - y^2 - z$$

   i) Find the parametric equations that represent the curve C. ii) Find the equation of the tangent line to the curve C at point $(-8, -4, 0)$. iii) Find the length of the curve from $(4, 0, 4)$ to $(-8, -4, 0)$. Use the calculator to evaluate the integral that you are going to get.

9. **Partial Derivatives.** Find the indicated derivatives.

   (a) $z = ax^2 e^{x^2 - xy}$ where a is a constant; z_x, z_y, z_{xx}, z_{xy} and z_{yy}.

   (b) $z = x \ln(xy^2)$; z_x, z_y, z_{xx}, z_{xy} and z_{yy}.

   (c) $xy^2 + yz^2 + zx^2 = 3$; z_x and z_y at $(1, 1, 1)$.

   (d) $x - yz = \cos(x + y + z)$; z_x and z_y at $(0, 1, -1)$.

10. **Tangent planes.** Find the equation of the tangent plane to a given surface at a specified point.

    (a) $z = y^2 - x^2$, at $(-4, 5, 9)$

    (b) $x^2 + 2y^2 + 3z^2 = 21$, at $(4, -1, 1)$

    (c) $xy^2 + yz^2 + zx^2 = 3$; at $(1, 1, 1)$.

11. **Linear Approximation.**

    (a) If $f(1, 2) = 3$, $f_x(1, 2) = 1$ and $f_y(1, 2) = -2$, approximate $f(.9, 1.99)$.

    (b) Find the linear approximation of $z = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate the value at $(1.95, 1.08)$.

12. **Applications.**

    (a) The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is rising at a rate of 0.15 K/s. The pressure P, volume V and temperature T are related by the equation $PV = 8.31T$. Find the rate of change of the volume when the pressure is 20 kPa and temperature 320 K.

(b) The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$ where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

(c) The number of flowers $N(S, T)$ in a closed environment depends on the amount of sunlight S that the flowers receive and the temperature T of the environment. Assume that $N_S = 2$ and $N_T = 4$. i) Assume that there are 100 flowers when $S = 50$ and $T = 70$. Use the linear approximation to estimate the number of flowers when $S = 52$ and $T = 73$. ii) If the temperature depends on time as $T(t) = 85 - 8/(1+t^2)$ and the amount of sunlight decreases on time as $S = 1/t$ find the rate of change of the flower population $N'(t)$ at time $t = 2$ days.

(d) Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

(e) Set up the equations for finding the dimensions of the rectangular box with the largest volume if the total surface area is 64 cm^2.

(f) A cardboard box without a lid is to have volume of 32,000 cm^3. Set up the equations for finding the dimensions that minimize the amount of cardboard used.

13. **Maximum and Minimum Values.** Find the maximum and minimum values of f.

(a) $z = 9 - 2x + 4y - x^2 - 4y^2$ (b) $z = x^2 + y^2 + x^2y + 4$

14. **Lagrange Multipliers.** Find the maximum and minimum values of f subject to the given constraint(s).

(a) $f(x, y) = x^2 - y^2$; $x^2 + y^2 = 1$ (b) $f(x, y, z) = 2x + 6y + 10z$; $x^2 + y^2 + z^2 = 35$

15. **Double Integrals.**

(a) $\int\int_D (x + 2y)dxdy$ where $D = \{ (x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq x^2 \}$

(b) $\int\int_D 2xdxdy$ where $D = \{ (x, y) \mid 0 \leq y \leq 1, \ y \leq x \leq e^y \}$

(c) $\int\int_D y^3 dxdy$ where D is the triangular region with vertices (0, 2), (1, 1) and (3, 2)

(d) Find the average value of the function $f(x, y) = 4x$ on the region D between the parabolas $y = x^2 - 2$ and $y = 3x - x^2$.

16. **The Volume.**

(a) Find the volume of the solid bounded by the plane $x + y + z = 1$ in the first octant.

(b) Find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$.

(c) Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$.

(d) Find the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

17. **Surface Area. Parametric Surfaces**

(a) Find the area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

(b) Find the area of the part of the surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(c) Find the equation of the tangent plane to the parametric surface $x = u + v$, $y = 3u^2$, $z = u - v$ at the point $(2, 3, 0)$.

(d) Find the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.

(e) Using the parametric equations and formula for the surface area for parametric curves, show that the surface area of the cylinder $x^2 + z^2 = 4$ for $0 \leq y \leq 5$ is 20π.

18. **Triple Integrals and volume.**

(a) $\int\int\int_E xy \; dx \; dy \; dz$ where E is the solid tetrahedron with vertices (0,0,0), (1, 0, 0), (0, 2, 0) and (0, 0, 3).

(b) $\int\int\int_E 2 \; dx \; dy \; dz$ where E is the solid that lies between the cylinders $x^2+y^2 = 1$ $x^2+y^2 = 4$ and between the xy-plane and the plane $z = x + 2$.

(c) $\int\int\int_E x^2 + y^2 + z^2 \; dx \; dy \; dz$ where E is the unit ball $x^2 + y^2 + z^2 \leq 1$.

(d) $\int\int\int_E z \; dx \; dy \; dz$ where E is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

(e) Find the volume of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$ by using the transformation $x = 2u$, $y = 3v$ $z = 5w$.

19. **Line Integrals.**

(a) $\int_C x \; y^4 \; ds$, C is the right half of the circle $x^2 + y^2 = 16$.

(b) $\int_C (xy + \ln x) \; dy$, C is the parabola $y = x^2$ from (1,1) to (3,9).

(c) $\int_C xy \; dx + (x - y) \; dy$, C consists of line segments from (0,0) to (2,0) and from (2,0) to (3,2).

(d) $\int_C z^2 \; dx + y \; dy + 2y \; dz$, where C consists of two parts C1 and C2. C1 is the intersection of the cylinder $x^2 + y^2 = 16$ and the plane $z = 3$ from $(0, 4, 3)$ to $(-4, 0, 3)$. C2 is a line segment from $(-4, 0, 3)$ to $(0, 1, 5)$.

(e) Find the work done by the force field $\overrightarrow{F}(x, y, z) = (x + y^2, y + z^2, z + x^2)$ in moving an object along the curve C which is the intersection of the plane $x + y + z = 1$ and the coordinate planes.

(f) Find the work done by the force field $\overrightarrow{F} = (-y^2, x, z^2)$ in moving an object along the curve C which is is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

20. **Potential. Independence of Path.**

(a) Check that $\vec{f} = \langle x^3y^4, x^4y^3 + 2y \rangle$ is conservative, find its potential function and use it to evaluate $\int_C \vec{f} d\vec{r}$ where C is $x = \sqrt{t}$, $y = 1 + t^3$, $0 \leq t \leq 1$.

(b) Check that $\vec{f} = \langle y, x+z, y \rangle$ is conservative, find its potential function and use it to evaluate $\int_C \vec{f} d\vec{r}$ where C is any path from $(2, 1, 4)$ to $(8, 3, -1)$.

(c) Show that the line integral $\int_C (2xy + z^2)dx + (x^2 + 2yz + 2)dy + (y^2 + 2xz + 3)dz$ where C is any path from (1, 0, 2) to (0, 1, 4), is independent of path and evaluate it.

21. **Green's Theorem.** Evaluate the following integrals using Green's theorem.

(a) $\oint_C x^4 dx + xy dy$ where C is the triangle with vertices (0, 0), (0, 1), and (1, 0). Compute the integral also without using Green's Theorem.

(b) $\oint_C y^2 dx + 3xy dy$ where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ above x-axis.

(c) $\oint_C e^y dx + 2xe^y dy$ where C is the square with vertices (0, 0), (1, 0), (1, 1), and (0, 1).

(d) $\oint_C xy dx + 2x^2 dy$ where C is the line segment from (-2,0) to (2,0) and the upper half of the circle $x^2 + y^2 = 4$.

22. **Curl and Divergence.** Find curl and divergence of the following vector fields.

(a) $\vec{f} = \langle xz, xyz, -y^2 \rangle$ (b) $\vec{f} = \langle e^x \sin y, e^x \cos y, z \rangle$

Solutions

More detailed solutions of the problems can be found on the class handouts.

1. Sequences. (a) convergent, limit is 0 (b) convergent, limit is 1/2 (c) convergent, limit is 3/2 (d) convergent, find limit from the equation $x = \frac{1}{1+x}$, the limit is .618 (e) convergent, find limit from the equation $x = \sqrt{2 + x}$, the limit is is 2 (f) Find limit from the equation $x^2 - 10^{-7}x - 10^{-14} = 0$. The relevant solution is $[H^+] = 1.618 \cdot 10^{-7}$ and pH= 6.7910.

2. Sum of Series. (a) $\frac{2^2}{3^{-1}} \sum_{n=2}^{\infty} \frac{2^n}{3^n} = \frac{12\frac{4}{9}}{\frac{1}{3}} = 16.$ (b) 4 (c) sum=3/2 (d) sum = 12/5

3. Convergence of Series.

(a) Divergent by the Divergence Test (b) Geometric Series. Convergent because 1/2 is between -1 and 1 (c) Divergent by the Divergence Test (d) p-series. Convergent because $4 > 1$ (e) p-series, $p = 3$. Convergent because $3 > 1$ (f) Convergent by the Integral Test (g) Note that the series is $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$. Use the Alternating Series Test with $b_n = \frac{1}{n}$. The sequence b_n has limit 0 and is decreasing. Thus, the series is convergent. (h) Divergent by the Divergence Test

(i) Convergent by the Alternating Series Test ($b_n = \frac{2n}{4n^2+1}$ has limit 0 and is decreasing)

(j) Note that the series is $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Use the Ratio Test. The limit from the test is $\frac{1}{2}$ which is less than 1 and so the series is convergent.

(k) Convergent by the Ratio Test (limit from the test is 0 which is less than 1)

(l) Convergent by the Root Test (limit from the test is 0 which is less than 1)

4. Convergence of Power Series. Series converges for (a) $-1/4 < x < 1/4$ (b) $-1 \le x < 3$ (c) All values of x (d) $-1/3 \le x < 1/3$ (e) $-1 \le x < 5$.

5. Power Series Expansion. (a) $e^{2x} = \sum_{n=0}^{\infty} \frac{2^n\, x^n}{n!}$ (b) $xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n\, x^{n+1}}{n!}$ (c) $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ (d) $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n\, x^n$ (e) $\sin 3x = 3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \frac{3^7x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$. (f) $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \Rightarrow \int_0^x e^{x^2} dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)}$.

6. Applications of Taylor Polynomials. (a) $1 + x + x^2/2 + x^3/6$. $e^{0.3} \approx 1.3495$ (b) $0 + x + 0 - x^3/6 + 0 = x - x^3/6$. $\sin(0.2) \approx .1987$ (c) $0 + x + x^2$. $e^{1/2}\sin(1/2) \approx .75$ (d) $f(2.1) \approx 5 + 3(.1) + 1/2(.1)^2 = 5.305$ (e) $f(1.9) \approx 5 + 3(-.1) + (1/2)(-.1)^2 + 1/12(-.1)^3 = 4.705$ (f) $f(1.01) \approx -1 - 1(.01) + 2/24(.01)^4 = -1.00999 \approx -1.01$

   (g) $f(v) = e^{\frac{hv}{kT}} - 1 \Rightarrow f'(v) = \frac{h}{kT}e^{\frac{hv}{kT}} \Rightarrow f''(v) = \frac{h^2}{k^2T^2}e^{\frac{hv}{kT}}$. Thus $f(0) = 1 - 1 = 0$, $f'(0) = \frac{h}{kT}$, $f''(0) = \frac{h^2}{k^2T^2}$. So $f(v) \approx \frac{hv}{kT} + \frac{h^2v^2}{2k^2T^2} = \frac{hv(2kT+hv)}{2k^2T^2}$.

   (h) $\frac{1}{(1-x)^2} \approx -1-2x-3x^2$ and $\frac{1}{(1+x)^2} \approx -1+2x-3x^2 \Rightarrow E = \frac{kq}{r^2}\left(-1 + 2\frac{d}{r} - 3\frac{d^2}{r^2} + 1 + 2\frac{d}{r} + 3\frac{d^2}{r^2}\right) = \frac{4kqd}{r^3}$.

7. Lines and Planes. (a) $x = 1 + t \quad y = 3t \quad z = 6 + t$ (b) $x = 3 \quad y = 1 + t \quad z = -1 - 5t$ (c) $3x - 7z = -9$ (d) $x + y + z = 2$

8. Curves in Space. (a) Tangent: $x = 1$, $y = t$, $z = 2 - t$. Length: 1.91 (b) i) $y = 4\cos t$, $z = 4\sin t$, $x = 8 - y^2 - z = 8 - 16\cos^2 t - 4\sin t$. ii) $(-8, -4, 0)$ corresponds to $t = \pi$. Plugging π in derivative gives you $\langle 4, 0, -4\rangle$. Tangent line: $x = 4t - 8$ $y = -4$ $z = -4t$. iii) $(4, 0, 4)$ corresponds to $t = \pi/2$ and $(-8, -4, 0)$ to $t = \pi$. The length is 14.515.

9. Partial Derivatives.

   (a) $z_x = 2axe^{x^2-xy} + ax^2e^{x^2-xy}(2x - y) = a(2x + 2x^3 - x^2y)e^{x^2-xy}$, $z_y = ax^2e^{x^2-xy}(-x) = -ax^3e^{x^2-xy}$. Then $z_{xx} = a(2+6x^2-2xy)e^{x^2-xy} + a(2x+2x^3-x^2y)e^{x^2-xy}(2x-y)$ and $z_{yy} = -ax^3e^{x^2-xy}(-x) = ax^4e^{x^2-xy}$. Differentiating z_x with respect to y get $z_{xy} = a(-x^2)e^{x^2-xy} + a(2x+2x^3-x^2y)e^{x^2-xy}(-x) = a(-x^2-2x^2-2x^4+x^3y)e^{x^2-xy} = a(-3x^2-2x^4+x^3y)e^{x^2-xy}$. Alternatively, differentiating z_y with respect to x get $z_{xy} = -3ax^2e^{x^2-xy} - ax^3e^{x^2-xy}(2x - y) = a(-3x^2 - 2x^4 + x^3y)e^{x^2-xy}$

   (b) $z_x = \ln(xy^2) + 1, \quad z_y = 2x/y, \quad z_{xx} = 1/x, \quad z_{xy} = z_{yx} = 2/y, \quad z_{yy} = -2x/y^2$

   (c) $z_x = -(y^2 + 2xz)/(2yz + x^2)$, $z_y = -(2xy + z^2)/(2yz + x^2)$. At $(1, 1, 1)$, $z_x = -1$, $z_y = -1$.

   (d) $z_x = -(1 + \sin(x + y + z))/(-y + \sin(x + y + z))$ and $z_y = -(-z + \sin(x + y + z))/(-y + \sin(x + y + z))$. At $(0, 1, -1)$, $z_x = 1$ and $z_y = 1$.

10. Tangent planes. (a) $8x + 10y - z = 9$ (b) $F_x = 2x, F_y = 4y, F_z = 6z$. At $(4, -1, 1)$ this produces vector $\langle 8, -4, 6\rangle$. The tangent plane is $4x - 2y + 3z = 21$. (c) $F_x = y^2 + 2xz, F_y = 2xy + z^2, F_z = 2yz + x^2$. At $(1, 1, 1)$ this produces vector $\langle 3, 3, 3\rangle$. The tangent plane $x + y + z = 3$.

11. Linear Approximation. (a) $f(.9, 1.99) \approx 2.92$ (b) $f(1.95, 1.08) \approx 2.847$

12. Applications.

    (a) -.27 liter per second (b) 2 degrees Celsius per second

    (c) i) 100+2(52-50)+4(73-70)=116 flowers. ii) $N_t = N_S S_t + N_T T_t = 2(-1/4) + 432/25 = 4.62$ flowers/day. (d) $(0, 0, \pm 1)$

    (e) Equations: $yz - 2\lambda y - 2\lambda z = 0$, $xz - 2\lambda x - 2\lambda z = 0$, $xy - 2\lambda x - 2\lambda y = 0$, $2xy + 2yz + 2xz = 64$. If solved, the equations would yield: $x = y = z = 4\sqrt{6}/3$ cm.

    (f) Equations: $y + 2z - yz\lambda = 0$, $x + 2z - xz\lambda = 0$, $2x + 2y - xy\lambda = 0$, $xyz = 32,000$. (If solved, the equations would yield: square base of side $x = y =$40 cm, height $z =$20 cm.)

13. Maximum and Minimum Values. (a) Maximum $f(-1, 1/2) = 11$ (b) Minimum $f(0, 0) = 4$, saddle points $(\pm\sqrt{2}, -1)$

14. Lagrange Multipliers. (a) Max. $f(\pm 1, 0) = 1$, min. $f(0, \pm 1) = -1$ (b) Max. $f(1, 3, 5) = 70$, min. $f(-1, -3, -5) = -70$

15. Double Integrals. (a) $\frac{9}{20}$ (b) 2.86 (c) $\frac{147}{20}$ (d) mass = 6, center of mass = $(\frac{3}{4}, \frac{3}{2})$

16. The Volume. (a) $\frac{1}{6}$ (b) $\frac{81\pi}{2}$ (c) $\frac{5\pi}{6}$ (d) 162π

17. Surface Area. Parametric Surfaces.

    (a) $3\sqrt{14}$ (b) 30.85 (c) Plane $3x - y + 3z = 3$

    (d) Parameterization: $x = r\cos t$, $y = r\sin t$, $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$. The length of the cross product is $\sqrt{2}r$. The surface area is $5\pi\sqrt{2}$.

    (e) Parameterization: $x = 2\cos t$, $y = y$, $z = 2\sin t$. Bounds: $0 \le t \le 2\pi$, $0 \le y \le 5$. Length of the cross product is 2. Thus the double integral is $2\pi \cdot 5 \cdot 2 = 20\pi$.

18. Triple Integrals and volume. (a) $\frac{1}{10}$ (b) 12π (c) $\frac{4\pi}{5}$ (d) $\frac{15\pi}{16}$ (e) 40π

19. Line Integrals. (a) 1638.4 (b) 102.68 (c) 17/3 (d) $\int_{C_1} = -44$, $\int_{C_2} = 67.83$. So, $\int_C = 67.83 - 44 = 23.83$

    (e) Let C_1 be a line from $(1, 0, 0)$ to $(0, 1, 0)$, C_2 a line from $(0, 1, 0)$ to $(0, 0, 1)$ and C_3 a line from $(0, 0, 1)$ to $(1, 0, 0)$. Find that $\int_{C_1} = \frac{-1}{3}$, $\int_{C_2} = \frac{-1}{3}$, and $\int_{C_3} = \frac{-1}{3}$. Thus $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -1$.

    (f) C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - y = 2 - \sin t$, $0 \le t \le 2\pi$. $\int_C = \int_C -y^2 dx + x dy + z^2 dz = \int_0^{2\pi} \sin^3 t dt + \cos^2 t dt + (2 - \sin t)^2 \cos t dt = \pi$.

20. Potential. Independence of Path. (a) $F = \frac{1}{4}x^4 y^4 + y^2 + c$, $\int_C = F(1, 2) - F(0, 1) = 7$ (b) $F = xy + yz + c$, $\int_C = F(8, 3, -1) - F(2, 1, 4) = 15$ (c) $F = x^2 y + z^2 x + y^2 z + 2y + 3z + c$, $\int_C = F(0, 1, 4) - F(1, 0, 2) = 8$.

21. Green's Theorem. (a) $\frac{1}{6}$ (b) $\frac{14}{3}$ (c) $e - 1$ (d) 0

22. Curl and Divergence. (a) $\text{div}\vec{f} = z + xz$, $\text{curl}\vec{f} = \langle -y(x+2), x, yz \rangle$ (b) $\text{div}\vec{f} = 1$, $\text{curl}\vec{f} = \langle 0, 0, 0 \rangle$