Calculus 3
Lia Vas

Sequences

A **sequence** is a list of numbers indexed by the positive (or nonnegative) integers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

A sequence might be given by the formula for its n-th term a_n. For example, $a_n = 1/n$. A sequence can also be given **recursively**. For example, $a_1 = 1$, $a_{n+1} = 1/(1 + a_n)$.

A sequence with the n-th term a_n is **convergent** if the limit $\lim_{n\to\infty} a_n$ exist (as a real number). Otherwise, it is **divergent.**

The limit $\lim_{n\to\infty} a_n$ exist and is equal to L if and only if $\lim_{x\to\infty} f(x)$ exist and is equal to L where $f(n) = a_n$. This enables us to use the **L'Hopital's Rule** when doing limits of sequences as well.

Practice Problems.

1. List five first terms of the sequence. Determine whether the above sequences are convergent or divergent. If they are convergent, find their limits.

   a) $a_n = (1/2)^n$, $n = 1, 2\ldots$ b) $a_n = \frac{n+1}{2n-1}$, $n = 1, 2\ldots$

   c) $a_n = \sin(n\pi/2)$, $n = 0, 1, 2\ldots$ d) $a_0 = 0$, $a_{n+1} = \sqrt{2 + a_n}$.

   e) $a_1 = 1$, $a_{n+1} = \frac{1}{1+a_n}$.

2. Determine whether the sequences are convergent or divergent. If they are convergent, find their limits.

   a) $a_n = \frac{\ln n}{n}$, $n = 1, 2\ldots$ b) $a_n = \frac{n}{2^n}$, $n = 0, 1, 2\ldots$ c) $a_n = \frac{n+3n^3}{4n^2+35n-7+2n^3}$, $n = 1, 2\ldots$

3. If the n-th term a_n is of the form $\frac{p_m(n)}{q_k(n)}$ where $p_m(x)$ is a polynomial with the leading term ax^m and $q_k(x)$ is a polynomial with the leading term bx^k, discuss the possible values of the limit $\lim_{n\to\infty} a_n$ considering the following three cases: (1) $m < k$, (2) $m = k$, (3) $m > k$.

4. Determine for which values of r is the sequence with the n-th term r^n, $n = 0, 1, 2\ldots$, convergent by discussing the following three cases (1) $-1 < r < 1$, (2) $r > 1$, (3) $r = 1$, (4) $r \leq -1$.

5. When calculating the hydrogen ion concentration $[H^+]$ in a acid-base system, the problem frequently boils down to finding the limit of a recursive sequence. For example, when hydrochloric acid HCl is dissolved in water, we have $[H^+]_1 = C_{HCl}$ and

$$[H^+]_{n+1} = C_{HCl} + \frac{K_w}{[H^+]_n},$$

   where C_{HCl} is the analytical concentration of HCl and K_w is the water's autoprotolysis constant that is equal to 10^{-14} at 25 degrees Centigrade. If the analytical concentration of HCL, C_{HCl}, is equal to 10^{-7}, find the hydrogen ion concentration $[H^+]$ and its pH value.

6. **Fibonacci numbers and the golden ratio.** Fibonacci numbers are terms of the following recursive sequence.

$$f_0 = 0,\ f_1 = 1 \quad \text{and} \quad f_{n+2} = f_{n+1} + f_n$$

for $n = 0, 1, \ldots$. Thus one starts with 0 and 1, and then produces the next Fibonacci number by adding the two previous Fibonacci numbers. The following sequence is obtained

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$

This sequence is clearly divergent.

On the other hand, two quantities are said to be in the **golden ratio** if the ratio of the larger to smaller quantity is the same as the ratio of their sum to the larger quantity. So, if a and b are two quantities and $a > b$, then a and b are in the golden ratio if

$$\frac{a}{b} = \frac{a+b}{a}.$$

To find the ratio $x = \frac{a}{b}$, note that the right side is $\frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{1}{x}$. Thus,

$$x = 1 + \frac{1}{x} \ \Rightarrow\ x^2 = x + 1 \ \Rightarrow\ x^2 - x - 1 = 0.$$

The positive solution of this quadratic equation is $\frac{1 \pm \sqrt{5}}{2} \approx 1.618$ prominently used in science as well as in art, architecture and music (you can explore Wikipedia to find out more).

While the Fibonacci sequence is divergent the **quotient of two consecutive terms of the Fibonacci sequence** $\frac{f_{n+1}}{f_n}$ is convergent. Show that its limit is the golden ratio.

Solutions.

1. a) First five terms: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. $\lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{\infty} = 0$. So, the sequence is convergent.

   b) First five terms: 2, 1, $\frac{4}{5}, \frac{5}{7}, \frac{6}{9}$. $\lim_{n\to\infty} \frac{n+1}{2n-1} = \lim_{n\to\infty} \frac{1}{2}$ (you can use L'Hopital's rule for example). So, the limit of the sequence is $\frac{1}{2}$ and so the sequence is convergent.

   c) First five terms: 0, 1, 0, -1. The terms of the sequence alternate between 0 and ± 1. Thus, there is not a single number towards which the terms converge. Thus, this sequence is divergent.

   d) First five terms: $0, \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$. Let a stand for the limit of this sequence in case it exists. Note that then $a = \lim_{n\to\infty} a_n$ and $a = \lim_{n\to\infty} a_{n+1}$ as well. To find the value of a let $n \to \infty$ in the equation $a_{n+1} = \sqrt{2 + a_n}$. The left side converges to a and the right side to $\sqrt{2+a}$. So, a can be found from the equation $a = \sqrt{2+a} \Rightarrow a^2 = 2 + a \Rightarrow a^2 - a - 2 = 0 \Rightarrow a = 2$ or $a = -1$. Since -1 is an extraneous root (it does not satisfy the equation $a = \sqrt{2+a}$), the limit of the sequence is 2.

   e) First five terms: 1, $\frac{1}{1+1}, \frac{1}{1+\frac{1}{1+1}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}$. Let a stand for the limit of this sequence in case it exists. Note that then $a = \lim_{n\to\infty} a_n$ and $a = \lim_{n\to\infty} a_{n+1}$ as well. To find the value of a let $n \to \infty$ in the equation $a_{n+1} = \frac{1}{1+a_n}$. The left side converges to a and the right side to $\frac{1}{1+a}$. So, a can be found from the equation $a = \frac{1}{1+a} \Rightarrow a(1+a) = 1 \Rightarrow a^2 + a - 1 = 0 \Rightarrow a = 0.618$ or $a = -1.618$. Since all the terms of the sequence are positive, the sequence converges towards the positive value $a = 0.618$.

2. a) $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = \frac{1}{\infty} = 0.$

   b) $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = \frac{1}{\infty} = 0.$

   c) Note that a_n is a quotient of two polynomials. Recall that the leading term of a polynomial determines it behavior at ∞. Thus $\lim_{n\to\infty} \frac{n+3n^3}{4n^2+35n-7+2n^3} = \lim_{n\to\infty} \frac{3n^3}{2n^3} = \frac{3}{2}.$

3. Since the leading term of a polynomial determines its behavior at infinity, the limit $\lim_{n\to\infty} \frac{p_m(n)}{q_k(n)}$ is equal to $\lim_{n\to\infty} \frac{an^m}{bn^k}$. So, if $m < k$ this limit is of the form $\frac{a}{bn^{k-m}} \to \frac{a}{\infty} = 0.$

   If $m = k$ this limit is of the form $\frac{an^m}{bn^m} = \frac{a}{b}.$

   if $m > k$ this limit is of the form $\frac{an^{m-k}}{b} \to \pm\infty$ (∞ if $\frac{a}{b}$ is positive and $-\infty$ if $\frac{a}{b}$ is negative).

4. (1) If $-1 < r < 1$, the powers of r approach 0 as n approaches infinity (consider, for example, what happens if $r = \frac{1}{2}$ or $r = \frac{-1}{2}$). (2) If $r > 1$, the powers of r become larger so the limit of the sequence is ∞. (3) If $r = 1$, the sequence is constant 1 and its limit is 1. (4) If $r \leq -1$, the terms alternate from positive to negative. So, the limit does not exist. In conclusion, the sequence is convergent just for $-1 < r \leq 1$.

5. the hydrogen ion concentration $[H^+]$ can be obtained as a limit of the recursive sequence. With the given values, the recursive equation becomes $[H^+]_1 = 10^{-7}$ and $[H^+]_{n+1} = 10^{-7} + \frac{10^{-14}}{[H^+]_n}$. Let x stand for the limit of this sequence. Then x can be found from the equation $x = 10^{-7} + \frac{10^{-14}}{x} \Rightarrow x^2 - 10^{-7}x - 10^{-14} = 0$. The positive solution $1.618 \cdot 10^{-7}$ gives us the value of $[H^+]$ and its pH value of 6.7910.

6. Let us denote $a_n = \frac{f_{n+1}}{f_n}$. Dividing the equation $f_{n+2} = f_{n+1} + f_n$ by f_{n+1}, we obtain $\frac{f_{n+2}}{f_{n+1}} = 1 + \frac{f_n}{f_{n+1}}$. Note that the term on the left is $a_{n+1} = \frac{f_{n+2}}{f_{n+1}}$ and that the right side is $1 + \frac{1}{a_n}$. Thus, the recursive formula of the quotient sequence a_n is $a_{n+1} = 1 + \frac{1}{a_n}$.

   Denote the limit by x. Thus $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and so value x satisfies the equation $x = 1 + \frac{1}{x}$. Multiply by x to get $x^2 = x + 1 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2} \approx 1.618$ and -0.618. Negative solution is not relevant since all terms of the sequence are positive. Thus, the limit is the golden ratio $\frac{1\pm\sqrt{5}}{2} \approx 1.618$.