Calculus 3
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Series

A **series** is a sum of numbers indexed by positive (or nonnegative) integers:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

For example, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

A series $\sum_{n=1}^{\infty} a_n$ is **convergent** if there is a real number a such that $\sum_{n=1}^{\infty} a_n = a$. Otherwise, it is **divergent.**

The sum s_n of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is called n**-th partial sum.** Note that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$$

Hence, $\sum_{n=1}^{\infty} a_n$ is convergent and has the sum a if and only if the sequence $\{s_n\}$ is convergent and has limit a.

The Divergence Test: If $\lim_{n\to\infty} a_n$ is not equal to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Intuitively, this statement establishes the fact that if the terms of the sequence are not small enough, their sum is not convergent. Note that the converse of this test is not true: there are divergent series whose n-th terms converge to 0. We shall see that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent regardless of the fact that $\frac{1}{n}$ converges to 0.

A **geometric series** is any series such that the ratio of two consecutive terms is constant r. In general, this series has the form

$$ar^k + ar^{k+1} + ar^{k+2} + \ldots = ar^k \sum_{n=0}^{\infty} r^n.$$

Let us concentrate on the series $\sum_{n=0}^{\infty} r^n$. By the divergent test and problem 4 from the last section, this series is divergent if $r \geq 1$ and $r \leq -1$ because in those cases the limit of r^n is not zero. It can be shown [1] that this series is convergent for $-1 < r < 1$ and has the sum

$$1 + r + r^2 + r^3 + \ldots = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Thus, the geometric series $\sum_{n=k}^{\infty} ar^n = ar^k + ar^{k+1} + ar^{k+2} + \ldots = ar^k \sum_{n=0}^{\infty} r^n$ has the sum

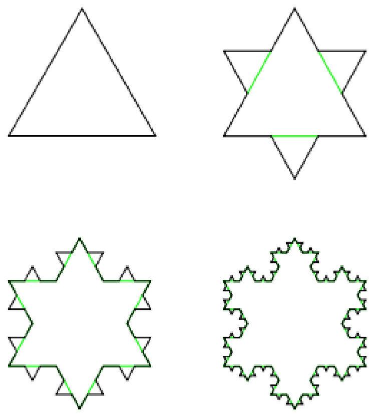
$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$$

[1] Assuming that the sum $1 + r + r^2 + \ldots$ is convergent and equal to s, then $rs = r + r^2 + r^3 + \ldots$ and so s and rs differ just in the first term 1 of s. So, $s = 1 + rs$. Solving this equation for s gives you that $s = \frac{1}{1-r}$.

Alternatively, note that $1 - r^n$ factors as $(1-r)(1 + r + \ldots + r^{n-1})$. So, the n-th partial sum $1 + r + r^2 + \ldots + r^{n-1}$ is equal to $\frac{1-r^n}{1-r}$. Since the sum of a series is the limit of its n-th partial sum, $\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1-0}{1-r}$ if $-1 < r < 1$. This gives you that the sum is $\frac{1}{1-r}$.

Geometric Series in Action

1. A fractal figure with finite area and infinite perimeter. Consider the following iterative construction.



Start with an equilateral triangle, then recursively altering each line segment as follows:

- Divide each side into three equal parts.
- Replace the middle part by an equilateral triangle pointing outwards with the middle part as its base and remove the base.

Assuming that we continue this construction indefinitely, we obtain a figure that is known as the **Koch snowflake**.

This was one of the first **fractal** curves considered. It is named after Niels Fabian Helge von Koch and the similar construction that appeared in his 1904 paper. Let us calculate the area and the perimeter of the Koch snowflake.

The number of sides and the triangles. For each iteration, *one side* of the figure from the previous stage becomes *four sides* in the following stage. Since we begin with three sides, the number of sides in first few consecutive steps are $3, 3(4), 3(4)^2, 3(4)^3, \ldots, 3(4)^n, \ldots$ So, the number of triangles added in each step is the same.

The area. Recall that the height of an equilateral triangle of side a is $h = a \sin 60° = \frac{a\sqrt{3}}{2}$. So, the area is $\frac{ah}{2} = \frac{a^2\sqrt{3}}{4}$.

Assume that we start with a triangle of side 1. The sides of the triangles forming the Koch snowflake have the lengths $1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^n}, \ldots$ so their areas are $\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4(9)}, \frac{\sqrt{3}}{4(9)^2}, \ldots, \frac{\sqrt{3}}{4(9)^n}, \ldots$ Thus, we have the following.

	step 0	step 1	step 2	step 3	...	step n
no. of sides	3	$3(4)$	$3(4)^2$	$3(4)^3$	...	$3(4)^n$
no. of new triangles	1	3	$3(4)$	$3(4)^2$	...	$3(4)^{n-1}$
length of each side	1	$\frac{1}{3}$	$\frac{1}{3^2}$	$\frac{1}{3^3}$	...	$\frac{1}{3^n}$
size of added area	$\frac{\sqrt{3}}{4}$	$3\frac{\sqrt{3}}{4(3)^2}$	$3(4)\frac{\sqrt{3}}{4(3^2)^2}$	$3(4)^2\frac{\sqrt{3}}{4(3^3)^2}$	...	$3(4)^{n-1}\frac{\sqrt{3}}{4(9)^n}$
perimeter	3	$3(4)\frac{1}{3}$	$3(4)^2\frac{1}{3^2}$	$3(4)^3\frac{1}{3^3}$	...	$3(4)^n\frac{1}{3^n}$

Multiplying those areas with the number of triangles we obtain the series computing the total area

$$A = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4(9)} + \frac{3(4)\sqrt{3}}{4(9)^2} + \frac{3(4)^2\sqrt{3}}{4(9)^3} + \ldots + \frac{3(4)^{n-1}\sqrt{3}}{4(9)^n} + \ldots$$

Note that all the terms except the first one follow the same pattern. Factoring $\sqrt{3}$ and $\frac{3}{4(9)} = \frac{1}{12}$ out of all the terms except the first one, we obtain

$$A = \sqrt{3}(\frac{1}{4} + \frac{1}{12}(1 + \frac{4}{9} + \frac{4^2}{9^2} + \ldots + \frac{(4)^{n-1}}{(9)^{n-1}} + \ldots) = \sqrt{3}(\frac{1}{4} + \frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n).$$

Thus, the area can be found using the formula for the sum of a geometric series with $r = \frac{4}{9}$. Note that $\frac{4}{9} < 1$ so this series is convergent. Using the formula obtain that

$$A = \sqrt{3}(\frac{1}{4} + \frac{1}{12}\frac{1}{1-\frac{4}{9}}) = \frac{2\sqrt{3}}{5} = 0.693.$$

The perimeter. The perimeter can be computed as the number of sides multiplied by the length of the side. At step n, the number of sides is $3(4)^n$ and the length of the side is $\frac{1}{3^n}$. So, the perimeter is

$$P = \lim_{n\to\infty} \frac{3(4)^n}{3^n} = 3 \lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty$$

since $\frac{4}{3}$ is larger than 1 so that the sequence $\left(\frac{4}{3}\right)^n$ is divergent. Thus, the infinite perimeter of the Koch snowflake encloses a finite area.

2. A Pharmacy Application. Assume that a person is given the same dose of a medicine at equally spaced time intervals. The body metabolizes some of the drug so that, after some time, only a certain percent of the original amount remains. After the each dose, the amount of the drug in the body is equal to the amount of the given dose plus the amount remnant from the previous doses. Let a be the amount given in each dose, and r the percent of the previous dose remaining in the body. Then, the amount A_n of the drug present after the n-th dose is

$$A_n = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

After a long time, the amount of the drug in the body present before and after the new dose stabilizes. The amount of the drug present after the new dose stabilizes at

$$A = a + ar + ar^2 + \ldots = \frac{a}{1-r}.$$

The amount A is called the **steady state**. The amount of the drug present before the new dose stabilizes at $A - a$.

A similar analysis can be used to compute the amount of herbicides or pesticides accumulated in living organisms, how long the natural resources will last assuming that the current usage levels increase at a constant rate and other phenomena. In business and economy, geometric series are used for computing the amount present in an account if the deposits are made repeatedly, annuities, market stabilization point etc.

Practice Problems.

1. Calculate the fifth partial sum of the following series. (a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

2. Determine whether the series are convergent or divergent. If they are convergent, find their sums.

   (a) $\sum_{n=1}^{\infty} \frac{3}{2^n}$ (b) $\sum_{n=2}^{\infty} \frac{2^{n+2}}{3^{n-1}}$

   (c) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ (d) $\sum_{n=1}^{\infty} \frac{n^2}{n^2+5}$

   (e) $\sum_{n=1}^{\infty} \frac{2^{2n}}{3^{n-1}}$ (f) $\sum_{n=1}^{\infty} \frac{n^2}{13n^2+4n+5}$

3. Represent the following decimal number as a quotient of integers (without using the calculator).

(a) $0.222222\ldots$ (b) $0.27272727\ldots$ (c) $1.2345454545\ldots$

(d) Find the sum $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \ldots$

(e) Find the sum $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \frac{3}{256} - \ldots$

4. Find the values of x for which the series converges. Find the sum for those values of x.

$$a)\ \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \qquad b)\ \sum_{n=1}^{\infty} 4^n x^n \qquad c)\ \sum_{n=0}^{\infty} \frac{3^{n+1}}{x^n}$$

5. Show that the series $\sum_{n=1}^{\infty} (-1)^n$ is divergent. Explain what is wrong with the following reasoning:

$$0 = (1-1) + (1-1) + (1-1) + \ldots = 1 + (-1+1) + (-1+1) + (-1+1) + \ldots = 1$$

Guido Ubaldus thought that he proved existence of God because "something was created out of nothing."

6. A person with an ear infection takes 200 mg ampicillin tablet once every 4 hours. About 12% of the drug in the body at the start of a four hour period is still there at the end of that period. Determine the quantity of ampicillin in the body (a) Right after taking the third tablet;

(b) Right after taking the sixth tablet; (c) In the long run right after taking a tablet;

(d) In the long run right before taking a tablet.

7. A person takes 100 mg of a drug at regular time intervals. About 15% of the drug in the body at the start of a new time period is still there at the end of that period. Determine the quantity of the drug in the body (a) right after taking the fourth dose; (b) in the long run right after taking a dose; (c) in the long run right before taking a dose.

8. Every day person consumes 5 micrograms of a toxin which leaves the body at a rate of 2% per day. Determine the amount of toxin that accumulated in the body in the long run.

Solutions.

1. (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = .969$ (b) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} = 3.55$

2. (a) $\sum_{n=1}^{\infty} \frac{3}{2^n} = \sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n$. So, this is a geometric series with $r = \frac{1}{2}$ and $a = 3$. Using the formula $\frac{ar^k}{1-r}$ with $k = 1$, we obtain that the sum is 3.

(b) $\sum_{n=2}^{\infty} \frac{2^{n+2}}{3^{n-1}} = \sum_{n=2}^{\infty} \frac{2^2 . 2^n}{3^{-1} . 3^n} = \sum_{n=2}^{\infty} \frac{2^2}{3^{-1}} \frac{2^n}{3^n} = \sum_{n=2}^{\infty} 12\left(\frac{2}{3}\right)^n$. So, this is a geometric series with $r = \frac{2}{3}$ and $a = 12$. Using the formula $\frac{ar^k}{1-r}$ with $k = 2$, we obtain that the sum is $\frac{12\frac{4}{9}}{\frac{1}{3}} = 16$.

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \ldots = 1.$

(d) Use the Divergence Test. Find the limit of the n-th term to be $\lim_{n\to\infty} \frac{n^2}{n^2+5} = \lim_{n\to\infty} \frac{n^2}{n^2} = 1$. Since this limit is not equal to 0, the series is divergent.

(e) $\sum_{n=1}^{\infty} \frac{2^{2n}}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{3^{-1}} \frac{4^n}{3^n} = \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n$. Thus, this is a geometric series with $r = \frac{4}{3}$ and $a = 3$. However, since $r = \frac{4}{3}$ is greater than 1, the series is divergent.

(f) Use the Divergence Test. Find the limit of the n-th term to be $\lim_{n\to\infty} \frac{n^2}{13n^2+4n+5} = \lim_{n\to\infty} \frac{n^2}{13n^2} = \frac{1}{13}$. Since this limit is not equal to 0, the series is divergent.

3. (a) $0.222222\ldots = 0.2 + 0.02 + 0.002 + \ldots = \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \ldots = \sum_{n=1}^{\infty} 2\left(\frac{1}{10}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a = 2$, $r = \frac{1}{10}$ and $k = 1$, we have that the sum is $\frac{\frac{2}{10}}{\frac{9}{10}} = \frac{2}{9}$.

   (b) $0.27272727\ldots = 0.27 + 0.0027 + 0.000027 + \ldots = \frac{27}{100} + \frac{27}{100^2} + \frac{27}{100^3} + \ldots = \sum_{n=1}^{\infty} 27\left(\frac{1}{100}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a = 27$, $r = \frac{1}{100}$ and $k = 1$, we have that the sum is $\frac{\frac{27}{100}}{\frac{99}{100}} = \frac{27}{99} = \frac{3}{11}$.

   (c) $1.2345454545\ldots = 1.23 + 0.0045 + 0.000045 + 0.00000045 + \ldots = 1.23 + \frac{45}{100^2} + \frac{45}{100^3} + \frac{45}{100^4} + \ldots = 1.23 + \sum_{n=2}^{\infty} 45\left(\frac{1}{100}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a = 45$, $r = \frac{1}{100}$ and $k = 2$, we have that the sum is $1.23 + \frac{\frac{45}{100^2}}{\frac{99}{100}} = \frac{123}{100} + \frac{45}{99(100)} = \frac{123(99)+45}{99(100)} = \frac{12222}{9900} = \frac{679}{550}$.

   (d) Note that the alternating sign of the terms of a series can be represented by $(-1)^n$ or $(-1)^{n+1}$ (just be careful about your initial value of n). In this case, the n-th term of the sum $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \ldots$ is of the form $(-1)^n \frac{2}{3^n}$ (note that the first term corresponds to $n = 0$). So, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{2}{3^n} = \sum_{n=0}^{\infty} 2\left(\frac{-1}{3}\right)^n$. This is a geometric series with $a = 2$, $r = \frac{-1}{3}$ and $k = 0$. So, the sum is $\frac{2}{1+\frac{1}{3}} = \frac{6}{4} = \frac{3}{2}$.

   (e) $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \frac{3}{256} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{3}{4^n} == \sum_{n=0}^{\infty} 3\left(\frac{-1}{4}\right)^n$. This is a geometric series with $a = 3$, $r = \frac{-1}{4}$ and $k = 0$. So, the sum is $\frac{3}{1+\frac{1}{4}} = \frac{12}{5}$.

4. (a) This is a geometric series with $r = \frac{x}{2}$. It is convergent for $-1 < r = \frac{x}{2} < 1 \Rightarrow -2 < x < 2$. The sum is $\frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$.

   (b) This is a geometric series with $r = 4x$. It is convergent for $-1 < 4x < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4}$. The sum is $\frac{4x}{1-4x}$.

   (c) This is a geometric series $\sum_{n=0}^{\infty} 3\left(\frac{3}{x}\right)^n$ with $r = \frac{3}{x}$. It is convergent for $-1 < \frac{3}{x} < 1 \Rightarrow \frac{3}{x} < 1$ and $\frac{3}{x} > -1 \Rightarrow \frac{3-x}{x} < 0$ and $\frac{3+x}{x} > 0 \Rightarrow (x > 3$ or $x < 0)$ and $(x > 0$ or $x < -3) \Rightarrow x > 3$ and $x < -3$. The sum is $\frac{3}{1-\frac{3}{x}} = \frac{3x}{x-3}$.

5. Use divergence test or geometric series test with $r = -1$ to conclude that the series is divergent. Therefore, the sum does not exist as a real number. Regrouping terms of a divergent series cannot give you a finite sum, so neither 0 nor 1 is the sum of this series.

6. $a = 200$ and $r = 0.12$. (a) Evaluate $\frac{a(1-r^3)}{1-r}$ (or $a + ar + ar^2$) to get $\frac{200(1-.12^3)}{.88} = 226.88$ mg.

   (b) $\frac{a(1-r^6)}{1-r} = \frac{200(1-.12^6)}{.88} = 227.27$ mg (c) $\frac{a}{1-r} = \frac{200}{.88} = 227.27$ mg; d) $227.27 - 200 = 27.27$ mg.

7. Here $a = 100$ and $r = .15$. (a) $\frac{a(1-r^4)}{1-r} = \frac{100(1-.15^4)}{.85} = 117.59$ mg (b) $\frac{a}{1-r} = \frac{100}{.85} = 117.65$ mg (c) 117.65-100=17.65 mg.

8. Note $r = .98$ here. The sum is $A = \frac{5}{1-.98} = 250$ micrograms.