Calculus 3
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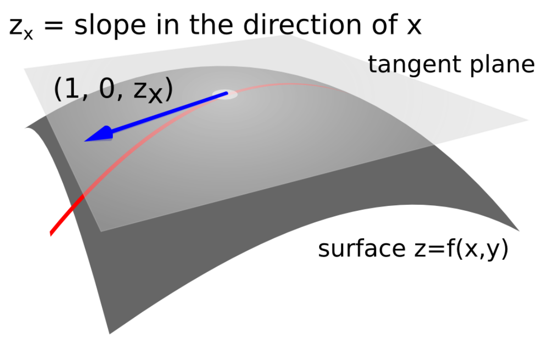
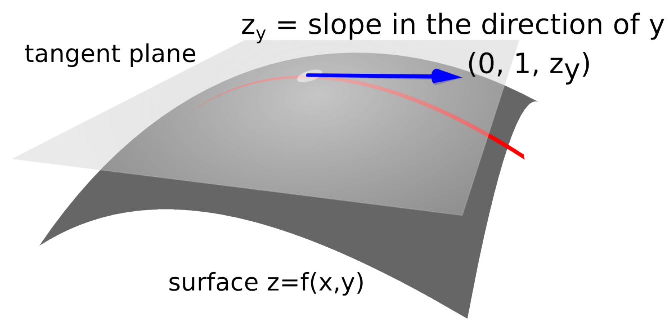
Tangent Plane. Linear Approximation. The Gradient

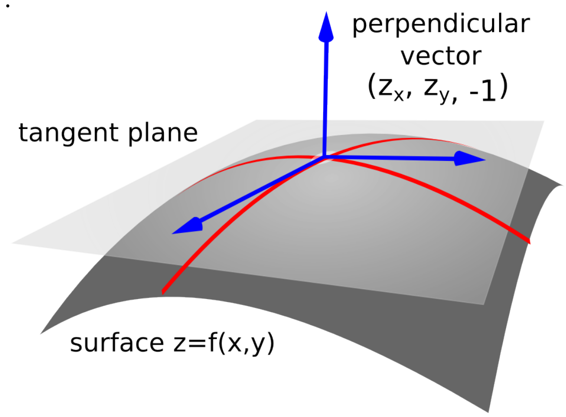
The tangent plane. Let $z = f(x, y)$ be a function of two variables with continuous partial derivatives. Recall that the vectors $\langle 1, 0, z_x\rangle$ and $\langle 0, 1, z_y\rangle$ are vectors in the tangent plane at any point on the surface.





Thus, the cross product of the vectors $\langle 1, 0, z_x\rangle$ and $\langle 0, 1, z_y\rangle$ is perpendicular to the tangent plane. Compute the cross product to be $\langle -z_x, -z_y, 1\rangle$.

So, vector $\langle -z_x, -z_y, 1\rangle$, its opposite $\langle z_x, z_y, -1\rangle$ or any of their multiples can be used as vectors for the tangent plane. In particular, an equation of **the tangent plane** of $z = f(x, y)$ at the point (x_0, y_0, z_0) can be obtained using (x_0, y_0, z_0) as point and $\langle z_x(x_0, y_0), z_y(x_0, y_0), -1\rangle$ as vector in the plane equation. This produces the equation



$$z_x(x_0, y_0)(x-x_0)+z_y(x_0, y_0)(y-y_0)-(z-z_0) = 0.$$

The linear approximation. Solving above equation for z produces the **linear approximation** of $z = f(x, y)$ with the tangent plane to point $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$.

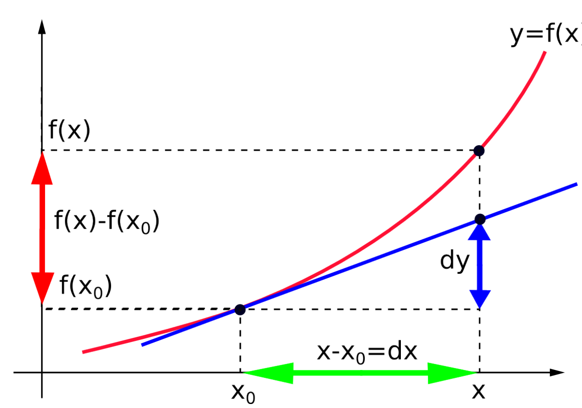
$$\begin{aligned} z = f(x, y) \approx \quad & z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = \\ & f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \end{aligned}$$

Compare this formula with the linear approximation formula from Calculus 1: if $y = f(x)$, the linear approximation of y with the tangent line to point $(x_0, f(x_0))$ is given by:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Also recall that you can think of this formula as follows.

$f(x)$	$\approx$	$f(x_0)$	$+$	$f'(x_0)$	$(x - x_0)$
future value	$\approx$	present value	$+$	change rate	time elapsed



In case when f is a function of two variables, the linear approximation formula can be interpreted as follows.

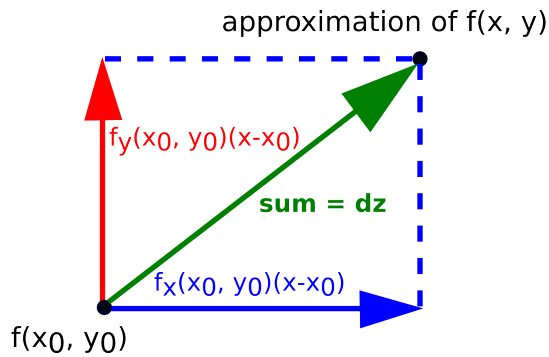
$f(x,y)$	$\approx$	$f(x_0,y_0)$	$+$	$f_x(x_0,y_0)$	$(x-x_0)$	$+$	$f_y(x_0,y_0)$	$(y-y_0)$
future value	$\approx$	present value	$+$	rate of change with respect to x	increment of x	$+$	rate of change with respect to y	increment of y

The Differential. The linear approximation formula can be also considered in the form

$$f(x,y)-f(x_0,y_0) \approx f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

The expression on the right measures the change in height between the surface and its tangent plane when (x_0,y_0) changes to (x,y). This quantity is called the **differential dz**. Thus,

$$dz = z_x dx + z_y dy$$



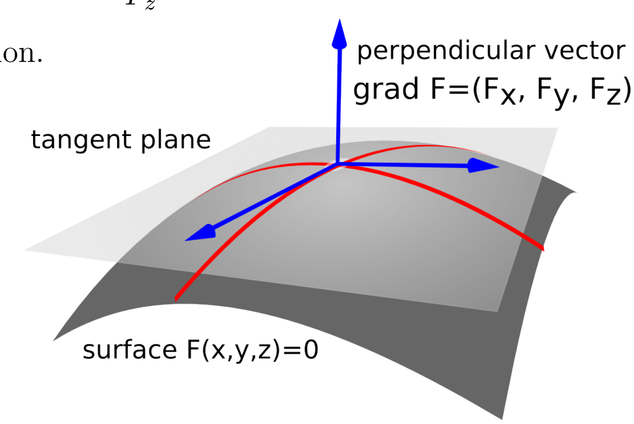
Implicit functions. In many situations, a surface can be given by an equation which cannot be solved for z. Spheres and cylinders are examples of this situation. In such cases, a surface is given by an **implicit function** $F(x,y,z)=0$ and the derivatives z_x and z_y cannot be found by direct differentiation but are given by the formulas

$$z_x = -\frac{F_x}{F_z} \quad \text{and} \quad z_y = -\frac{F_y}{F_z}$$

which validity will be demonstrated in the next section.

The vector $\langle z_x, z_y, -1\rangle$ used as a vector perpendicular to the tangent plane in this case becomes $\langle -\frac{F_x}{F_z}, -\frac{F_y}{F_z}, -1\rangle$. Scaling this vector by $-F_z$ (multiplying each coordinate by $-F_z$) produces the vector

$$\langle F_x, F_y, F_z\rangle$$



which is still perpendicular to the tangent plane.

Thus, the equation of the **tangent plane** of surface $F(x,y,z)=0$ at (x_0,y_0,z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

The vector $\langle F_x, F_y, F_z\rangle$ is called **the gradient** of function F. It has its two dimensional version as well.

Let $f(x,y)$ be a function of two variables. The **gradient** of f is the vector $\nabla f = \langle f_x, f_y \rangle$ sometimes also denoted by grad$f$.

The **gradient operator** is defined as
$\nabla = \langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y} \rangle$

Thus, $\nabla f = \langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y} \rangle = \langle f_x,\ f_y \rangle$

Let $F(x,y,z)$ be a function of three variables. The **gradient** of F is the vector $\nabla F = \langle F_x, F_y, F_z \rangle$ sometimes also denoted by grad$F$.

The **gradient operator** is defined as
$\nabla = \langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \rangle$

Thus, $\nabla F = \langle \frac{\partial F}{\partial x},\ \frac{\partial F}{\partial y},\ \frac{\partial F}{\partial z} \rangle = \langle F_x,\ F_y,\ F_z \rangle$

Using the gradient, a vector equation of the tangent plane of an implicit function F can be written as $\nabla F(\overrightarrow{r_0}) \cdot (\overrightarrow{r} - \overrightarrow{r_0}) = 0$.

Practice Problems.

1. Find an equation of the tangent plane to a given surface at the specified point.

   (a) $z = y^2 - x^2$, $(-4,5,9)$ (b) $z = e^x \ln y$, $(3,1,0)$

2. (a) If $f(2,3) = 5$, $f_x(2,3) = 4$ and $f_y(2,3) = 3$, approximate $f(2.02, 3.1)$.

   (b) If $f(1,2) = 3$, $f_x(1,2) = 1$ and $f_y(1,2) = -2$, approximate $f(.9, 1.99)$.

3. Find the linear approximation of the given function at the specified point.

   (a) $z = \sqrt{20 - x^2 - 7y^2}$ at $(2,1)$. Using the linear approximation, approximate the value at $(1.95, 1.08)$.

   (b) $z = \ln(x - 3y)$ at $(7,2)$. Using the linear approximation, approximate value at $(6.9, 2.06)$.

4. The number N of bacteria in a culture depends on temperature T and pressure P and it changes at the rates of 3 bacteria per kPa and 5 bacteria per Kelvin. If there is 300 bacteria initially when $T = 305$ K and $P = 102$ kPa, estimate the number of bacteria when $T = 309$ K and $P = 100$ kPa.

5. The number of flowers N in a closed environment depends on the amount of sunlight S that the flowers receive and the temperature T of the environment. Assume that the number of flowers changes at the rates $N_S = 2$ and $N_T = 4$. If there are 100 flowers when $S = 50$ and $T = 70$, estimate the number of flowers when $S = 52$ and $T = 73$.

6. Find the derivatives z_x and z_y of the following surfaces at the indicated points.

   (a) $x^2 + z^2 = 25$, $(-4,5,3)$ (b) $y^2 \sin z = xz^2 + 3ze^y$, $(-3,0,1)$

7. Find an equation of the tangent plane to a given surface at the specified point.

   (a) $x^2 + 2y^2 + 3z^2 = 21$, $(4,-1,1)$ (b) $xe^{yz} = z$, $(5,0,5)$

8. Find the gradient vector field of f for (a) $f(x,y) = \ln(x + 2y)$ (b) $f(x,y,z) = x^2 + y^2 + z^2$.

Solutions.

1. (a) $z_x = -2x$, $z_y = 2y$. When $x = -4$ and $y = 5$, $z_x = 8$ and $z_y = 10$. Thus $\langle z_x, z_y, -1\rangle = \langle 8, 10, -1\rangle$. With point $(-4, 5, 9)$, and vector $\langle 8, 10, -1\rangle$ obtain the plane $8(x+4)+10(y-5)-1(z-9) = 0 \Rightarrow 8x + 10y - z = 9$.

   (b) $z_x = e^x \ln y$, $z_y = \frac{e^x}{y}$. When $x = 3$ and $y = 1$, $z_x = 0$ and $z_y = e^3$. Using vector $\langle 0, e^3, -1\rangle$ and point $(3, 1, 0)$, obtain the tangent plane as $0(x-3)+e^3(y-1)-1(z-0) = 0 \Rightarrow e^3y-z = e^3$.

2. (a) $f(2.02, 3.1) \approx f(2, 3) + f_x(2, 3)(2.02 - 2) + f_y(2, 3)(3.1 - 3) = 5 + 4(0.02) + 3(0.1) = 5 + 0.08 + 0.3 = 5.38$
   (b) $f(.9, 1.99) \approx f(1, 2) + f_x(1, 2)(0.9 - 1) + f_y(1, 2)(1.99 - 2) = 3 + 1(-0.1) - 2(-0.01) = 3 - 0.1 + 0.02 = 2.92$

3. (a) Let $z = f(x, y)$. Find that $f(2, 1) = \sqrt{20 - 4 - 7} = \sqrt{9} = 3$. Then find $z_x = \frac{1}{2}(20 - x^2 - 7y^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{20-x^2-7y^2}}$, $z_y = \frac{1}{2}(20 - x^2 - 7y^2)^{-1/2}(-14y) = \frac{-7y}{\sqrt{20-x^2-7y^2}}$ and plug that $x = 2, y = 1$. Obtain that $f_x(2, 1) = \frac{-2}{3}$ and $f_y(2, 1) = \frac{-7}{3}$. Thus, $f(1.95, 1.08) \approx f(2, 1) + f_x(2, 1)(1.95 - 2) + f_y(2, 1)(1.08 - 1) = 3 - \frac{2}{3}(-0.05) - \frac{7}{3}(0.08) = 2.847$.

   (b) Let $z = f(x, y)$. Find that $f(7, 2) = \ln(x - 3y) = \ln(7 - 6) = \ln 1 = 0$. Then find $z_x = \frac{1}{x-3y}$, $z_y = \frac{-3}{x-3y}$ and plug that $x = 7, y = 2$. Obtain that $f_x(7, 2) = \frac{1}{1} = 1$ and $f_y(7, 2) = \frac{-3}{1} = -3$. Thus, $f(6.9, 2.06) \approx f(7, 2) + f_x(7, 2)(6.9 - 7) + f_y(7, 2)(2.06 - 2) = 0 + 1(-0.1) - 3(0.06) = -0.1 - 0.18 = -0.28$.

4. Let us denote the initial conditions of 305 K and 102 kPa by T_0 and P_0 so that $N(305, 102) = 300$. Let us denote the two given rates by N_P and N_T so that $N_P = 3$, and $N_T = 5$. Using the linear approximation formula, $N(T, P) \approx N(T_0, P_0) + N_T \cdot (T - T_0) + N_P \cdot (P - P_0)$ with $T = 309$ and $P = 100$, we have that $N(309, 100) \approx N(305, 102)+5(309-305)+3(100-102) = 300 + 5(4) + 3(-2) = 314$ bacteria.

5. Let us use the notation S_0 and T_0 for the initial conditions of 50 for S and 70 for T. Thus $N(50, 70) = 100$. Using the linear approximation formula and the fact that $N_S = 2$ and $N_T = 4$, we have that $N(S, T) \approx N(S_0, T_0) + N_S \cdot (S - S_0) + N_T \cdot (T - T_0) \Rightarrow N(52, 73) \approx N(50, 70) + 2(52 - 50) + 4(73 - 70) = 100 + 2(2) + 4(3) = 116$ flowers.

6. (a) Consider $F(x, y, z) = x^2 + z^2 - 25$. Then $F_x = 2x, F_y = 0$ and $F_z = 2z$. At $(-4, 5, 3)$, $F_x = -8, F_y = 0$ and $F_z = 6$. So, $z_x = -\frac{F_x}{F_z} = -\frac{-8}{6} = \frac{4}{3}$ and $z_y = -\frac{F_y}{F_z} = -\frac{0}{6} = 0$.

   (b) Consider $F = y^2 \sin z - xz^2 - 3ze^y$. Then $F_x = -z^2, F_y = 2y \sin z - 3ze^y$ and $F_z = y^2 \cos z - 2xz - 3e^y$. At $(-3, 0, 1)$, $F_x = -1, F_y = -3$ and $F_z = 6 - 3 = 3$. So, $z_x = -\frac{F_x}{F_z} = -\frac{-1}{3} = \frac{1}{3}$ and $z_y = -\frac{F_y}{F_z} = -\frac{-3}{3} = 1$.

7. (a) Consider $F = x^2 + 2y^2 + 3z^2 - 21 = 0$. Find $F_x = 2x$, $F_y = 4y$, and $F_z = 6z$. At $(4, -1, 1)$, $F_x = 8$, $F_y = -4$, and $F_z = 6$. Using vector $\langle 8, -4, 6\rangle$ and point $(4, -1, 1)$, obtain the equation of the plane $8(x - 4) - 4(y + 1) + 6(z - 1) = 0 \Rightarrow 8x - 4y + 6z = 42 \Rightarrow 4x - 2y + 3z = 21$.

   (b) Consider $F = xe^{yz} - z = 0$. Then $F_x = e^{yz}$, $F_y = xze^{yz}$, and $F_z = xye^{yz} - 1$. At $(5, 0, 5)$, $F_x = 1$, $F_y = 25$, and $F_z = -1$. Using vector $\langle 1, 25, -1\rangle$ and point $(5, 0, 5)$, obtain the tangent plane $1(x - 5) + 25(y - 0) - 1(z - 5) = 0 \Rightarrow x + 25y - z = 0$.

8. (a) $\langle f_x,\ f_y\rangle = \langle \frac{1}{x+2y}, \frac{2}{x+2y}\rangle$ (b) $\langle F_x,\ F_y,\ F_z\rangle = \langle 2x, 2y, 2z\rangle$.