Calculus 3
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Taylor Series and Polynomials

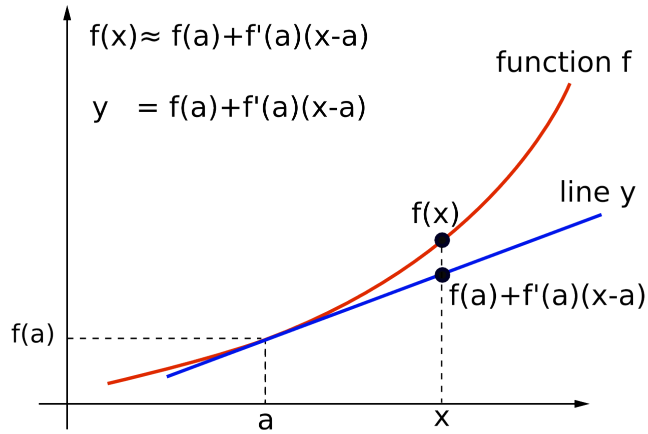
Taylor polynomial. Recall that the line which approximates a function $f(x)$ at a point $(a, f(a))$ has the slope $f'(a)$. By point-slope equation, the equation of this line is

$$y - f(a) = f'(a)(x-a) \Rightarrow y = f(a) + f'(a)(x-a).$$

The expression $f(a)+f'(a)(x-a)$ is the **linear approximation** of $f(x)$ at $x = a$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

Note that the **function value and the value of the first derivative** is the same for a function and its linear approximation.



In applications, you can think of the value $f(a)$ as of the **present value**, the value $f(x)$ then represents the **future value**, $(x-a)$ the **time lapsed** and $f'(a)$ the **change rate**. Thus

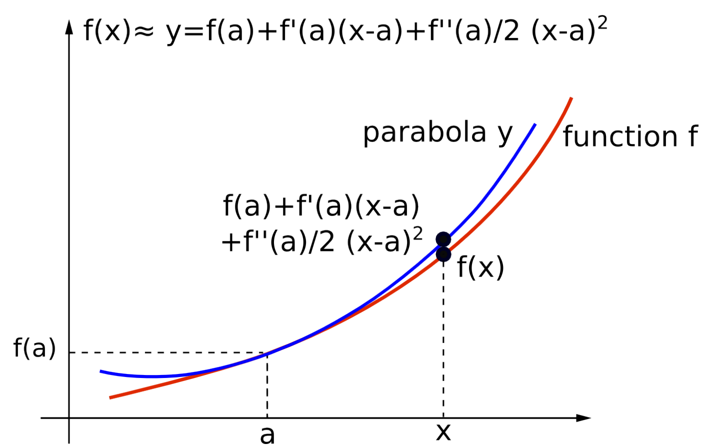
$f(x)$	$\approx$	$f(a)$	$+$	$f'(a)$	$(x-a)$
future value	$\approx$	present value	$+$	change rate	time elapsed

Assume now that we want to increase the accuracy of the approximation by approximating the function with a parabola $y = a_2(x-a)^2 + a_1(x-a) + a_0$ in such a way that the **function value and the value of the first and the second derivatives** are the same for $f(x)$ and parabola y when $x = a$. The condition $f(a) = y(a)$ implies that $a_0 = f(a)$. The condition that $f'(a) = y'(a)$ implies that $f'(a) = 2a_2(x-a) + a_1|_{x=a} = 0 + a_1$ so that $a_1 = f'(a)$.

The condition $f''(a) = y''(a)$ implies that $f''(a) = 2a_2 \Rightarrow a_2 = \frac{f''(a)}{2}$.

This produces the formula for polynomial of second degree that approximates the function $f(x)$ at $x = a$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$



Similarly, if we are to approximate the formula of a polynomial of third degree $y = a_3(x-a)^3 + a_2(x-a)^2 + a_1(x-a) + a_0$ which approximates function $f(x)$ at $x = a$, we obtain that $a_0 = f(a)$, $a_1 = f'(a)$, $a_2 = \frac{f''(a)}{2}$, and, equating $f'''(a)$ with $y'''(a) \Rightarrow f'''(a) = 2 \cdot 3a_3 = 6a_3$ so that $a_3 = \frac{f'''(a)}{6}$.

Note that the values in the denominators of the expressions $a_0 = f(a) = \frac{f(a)}{1}$, $a_1 = \frac{f'(a)}{1}$, $a_2 = \frac{f''(a)}{2}$, and $a_3 = \frac{f'''(a)}{6}$, match the values of the *factoriel* function of $0, 1, 2$ and 3. Recall that the product $1 \cdot 2 \cdot \ldots n$ is written shortly as $n!$ and is called the **factoriel** of n. Thus $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ and so on. $0!$ is defined to be 1.

Continuing in this way, we obtain the formula for approximating $f(x)$ with a polynomial of n-th degree to be

$$f(x) \approx f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$$

The polynomial on the right is called the **Taylor polynomial of $f(x)$ at $x = a$ of order n.** The Taylor polynomial centered at 0 is sometimes called **Maclaurin polynomial**.

Approximating a function using its Taylor polynomial is particularly useful when certain phenomena is modeled by a function which is either

- too complex to be manipulated or
- such that its exact formulas is not known.

but its value and the value of its derivatives are known at a point. Calculators and software applications (including Matlab for example) manipulate many functions using their Taylor polynomials.

Taylor Series. If a function $f(x)$ can be expressed as a power series $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ centered at a, continuing the process of equating value of the derivatives of function and the power series at $x = a$ as above, we obtain that $a_n = \frac{f^{(n)}(a)}{n!}$. Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This series is called the **Taylor series for $f(x)$ centered at a.** The sum of the first $n+1$ terms of the Taylor series is the Taylor polynomial of n-th degree at $x = a$.

Differentiation and Integration of Power Series. For the x-values in the interval of convergence of the power series $\sum_{n=1}^{\infty} a_n(x-a)^n$, you can differentiate and integrate the series by differentiating and integrating each term.

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} a_n(x-a)^n\right) = \sum_{n=1}^{\infty} a_n n(x-a)^{n-1}$$

$$\int_a^x \left(\sum_{n=1}^{\infty} a_n(x-a)^n\right) dx = \sum_{n=1}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$$

Using this fact, the power series can also be used for evaluating integrals which cannot be expressed in terms of elementary functions (i.e. functions whose antiderivatives are not elementary functions). Examples of such integrals include $\int e^{x^2} dx$, $\int \frac{\sin x}{x} dx$, $\int \frac{e^x}{x} dx$ etc.

Example. Find the power series expansion centered at 0 for e^x. Use the Taylor polynomial of degree 4 to approximate the value of e with a fraction. Compare with the calculator answer.

If $f(x) = e^x$, then $f' = f'' = f''' = \ldots = e^x$. The value of f and its derivatives at 0 is 1 and so $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$.

The Taylor polynomial of degree 4 is $p_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$. The x-value 1 corresponds to the y-value e of e^x. So, e^x is can be approximated by $p_4(1) = 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \frac{1^4}{24} = \frac{48+12+4+1}{24} = \frac{65}{24}$. Note that $e = 2.718...$ and that $\frac{65}{24} = 2.708...$ So, using the polynomial approximation of just degree 4, the approximation of e has the first two digits matching the exact value.

Practice Problems.

1. Find the power series expansion of the following functions centered at given point.

   (a) $x^2 - 2x + 1; \quad x = 0$ (b) $x^2 - 2x + 1; \quad x = 1$ (c) $e^x; \quad x = 1$

   (d) $e^{2x}; \quad x = 0$ (e) $xe^{2x}; \quad x = 0$ (f) $\frac{1}{1-x}; \quad x = 0$

   (g) $\frac{1}{1-x^2}; \quad x = 0$ (h) $\frac{1}{1+x}; \quad x = 0$ (i) $\frac{x}{1+x^2}; \quad x = 0$

   (j) $\sin x; \quad x = 0$ (k) $\sin 3x; \quad x = 0$ (l) $\cos x; \quad x = 0$

2. Find the power series expansion of the following functions centered at given point.

   (a) $\frac{1}{(1-x)^2}; \quad x = 0$ (b) $\ln(1-x); \quad x = 0$

3. Evaluate the following integrals as infinite series.

   (a) $\int_0^x e^{x^2}\,dx$ (b) $\int_0^x \frac{\sin x}{x}\,dx$

4. Find the Taylor polynomial of the given degree n centered at given point a for function $f(x)$.

   (a) $f(x) = e^x; \quad a = 0; \quad n = 4$. Use to approximate e with a rational number.
   (b) $f(x) = e^x; \quad a = 1; \quad n = 4$.
   (c) $f(x) = \sin x; \quad a = 0; \quad n = 4$. Use to approximate $\sin(.2)$ with a rational number.
   (d) $f(x) = e^x \sin x; \quad a = 0; \quad n = 3$. Use to approximate $e^{1/2} \sin \frac{1}{2}$ with a rational number.

5. (a) If $f(2) = 5$, $f'(2) = 3$ and $f''(2) = 1$, approximate $f(2.1)$.
   (b) If $f(2) = 5$, $f'(2) = 3$, $f''(2) = 1$, and $f'''(x) = \frac{1}{2}$ for all x, approximate $f(1.9)$.
   (c) If $f(1) = f'(1) = -1$, $f''(1) = f'''(1) = 0$ and $f^{iv}(1) = 2$, approximate $f(1.01)$.

6. ("PChem problem") Approximate the function $e^{\frac{hv}{kT}} - 1$ by its Taylor polynomial of the second degree in terms of v.

7. (“ Physics problem”) The magnitude of the electric field E of a single charge q can be described by $E = \frac{kq}{r^2}$ where r is the distance between the field and the charge and k is a proportionality constant. If two opposite charges on distance d from each other create an electric dipole moment, this formula changes to

$$E = \frac{kq}{(r-d)^2} - \frac{kq}{(r+d)^2} = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2}$$

Use the Taylor polynomial of the second degree of the function $f(x) = \frac{1}{(1-x)^2}$ (see problem 2 (a)) to show that the magnitude of the electric field E can be approximated as

$$E \approx \frac{4kqd}{r^3}$$

This approximation is accurate if r is much larger than d so that the quotient $\frac{d}{r}$ is small.

Solutions.

1. (a) Let $f(x) = x^2 - 2x + 1$. Then $f(0) = 1$, $f'(0) = -2$ and $f''(0) = 2$. All the other derivatives are 0. So, the Taylor series is $1 - 2x + \frac{2}{2}x^2 + 0 + 0 + \ldots = 1 - 2x + x^2$. Note that this is the same polynomial as $f(x)$.

   This answer should not be surprising. In fact, any polynomial is equal to its Taylor series expansion centered at 0.

   (b) Let $f(x) = x^2 - 2x + 1$. Then $f(1) = 0 = f'(1)$ and $f''(1) = 2$. So, the Taylor series expansion centered at 1 is $0 + 0 + \frac{2}{2}(x-1)^2 + 0 + 0 + \ldots = (x-1)^2$. Note that $(x-1)^2$ foils as $x^2 - 2x + 1$.

   (c) Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ for any n and $f^{(n)}(1) = e$. So, $e^x = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

   (d) Use that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. To get the expansion for e^{2x}, substitute x with $2x$ in the expansion of e^x. Thus $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

   (e) To get the expansion for xe^{2x}, multiply the expansion for e^{2x} from the previous problem by x. So, $xe^{2x} = x\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$.

   (f) Let $f(x) = \frac{1}{1-x}$. Then $f' = \frac{1}{(1-x)^2}$, $f'' = \frac{2}{(1-x)^3}$, $f''' = \frac{2\cdot 3}{(1-x)^4}$, ... $f^{(n)} = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot n}{(1-x)^{n+1}} = \frac{n!}{(1-x)^{n+1}}$. So $f^{(n)}(0) = n!$ and hence the expansion is $\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$.

   Alternatively, note that the formula for the sum of geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ gives the same answer if you put x for r.

   (g) To get the expansion for $\frac{1}{1-x^2}$, substitute x^2 for x in the expansion for $\frac{1}{1-x}$. So, $\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n}$.

   (h) To get the expansion for $\frac{1}{1+x}$, substitute $-x$ for x in the expansion for $\frac{1}{1-x}$. So, $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$.

   (i) To get the expansion for $\frac{x}{1+x^2}$, substitute x^2 for x in the expansion for $\frac{1}{1+x}$ and multiply it by x So, $\frac{x}{1+x^2} = x\sum_{n=0}^{\infty}(-1)^n(x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}$.

(j) Note that $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x \Rightarrow f'''(x) = -\cos x \Rightarrow f^{(4)}(x) = \sin x \Rightarrow f^{(5)}(x) = \cos x$ and the cycle continues. The values at 0 are: 0, 1, 0, -1, 0 ,1 ... So, all the terms with even power of x have zero coefficient. The expansion is $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

(k) To get the expansion for $\sin 3x$, substitute $3x$ for x in the expansion of $\sin x$. Hence $\sin 3x = 3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \frac{3^7x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$.

(l) Similar to finding expansion for $\sin x$. In this case, all the terms with odd power of x have zero coefficient. The expansion is $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

2. (a) Let $f(x) = \frac{1}{(1-x)^2}$. Note that $f(x)$ is the derivative of function $g(x) = \frac{1}{1-x}$ (since $g'(x) = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} = f(x)$) for which we already have the power series expansion. So, instead of finding the expansion of f using the formula for Taylor series, it is more efficient to differentiate the series for $g(x) = \frac{1}{1-x}$. So, $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow f(x) = g'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} nx^{n-1}$.

   (b) Let $f(x) = \ln(1-x)$. Then $f'(x) = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n \Rightarrow f(x) = \int_0^x \left(-\sum_{n=0}^{\infty} x^n\right) dx = -\sum_{n=0}^{\infty} \int_0^x x^n dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.

3. (a) Find the power series expansion of e^{x^2} using the expansion of e^x first. $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. Then integrate the power series to get $\int_0^x e^{x^2} dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)}$.

   (b) Divide the expansion for $\sin x$ by x to obtain the expansion for $\frac{\sin x}{x}$ and then integrate term by term. So, $\int_0^x \frac{\sin x}{x}\, dx = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n \int_0^x x^{2n} dx}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)} = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \ldots$

4. (a) The Taylor polynomial is $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$. Then substitute 1 for x to approximate $e = e^1$. Obtain $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{24+24+12+4+1}{24} = \frac{65}{24} = 2.7083$. Comparing with the calculator answer $e \approx 2.7183$, you can see that just first five terms compute the first two digits correctly.

   (b) $e^x \approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4$

   (c) The Taylor polynomial is $\sin x \approx x - \frac{x^3}{6}$ Substitute $0.2 = \frac{1}{5}$ for x to approximate $\sin \frac{1}{5}$. Obtain $\frac{1}{5} - \frac{1}{750} = \frac{149}{750} \approx .198666...$ Comparing with the calculator answer $\sin \frac{1}{5} \approx .198669$, you can see that the first five decimals are correct.

   (d) Let $f(x) = e^x \sin x$. Find the first three derivatives $f'(x) = e^x \sin x + e^x \cos x \Rightarrow f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x \Rightarrow f'''(x) = 2e^x \cos x - 2e^x \sin x$. Plug 0 and get $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$ and $f'''(0) = 2$. Thus $e^x \sin x \approx 0 + 1x + \frac{2}{2}x^2 + \frac{2}{6}x^3 = x + x^2 + \frac{x^3}{3}$. To approximate $e^{1/2} \sin \frac{1}{2}$ substitute $\frac{1}{2}$ for x in the Taylor polynomial. Obtain $\frac{1}{2} + \frac{1}{4} + \frac{1}{24} = \frac{19}{24} \approx .7917$. Compare with the calculator answer $e^{1/2} \sin \frac{1}{2} \approx .7904$.

5. (a) Let $x = 2.1$ and $a = 2$. The Taylor polynomial of degree 2 gives you $f(2.1) \approx f(2) + f'(2)(2.1-2) + \frac{f''(2)}{2}(2.1-2)^2 = 5 + 3(.1) + \frac{1}{2}(.1)^2 = 5.305$.

   (b) Let $x = 1.9$ and $a = 2$. Then $f(1.9) \approx f(2) + f'(2)(1.9-2) + \frac{f''(2)}{2}(1.9-2)^2 + \frac{f'''(2)}{6}(1.9-2)^3 = 5 + 3(-.1) + \frac{1}{2}(-.1)^2 + \frac{1}{12}(-.1)^3 = 4.705$.

(c) Let $x = 1.01$ and $a = 1$. Then $f(1.01) \approx f(1) + f'(1)(1.01-1) + \frac{f''(1)}{2}(1.01-1)^2 + \frac{f'''(1)}{6}(1.01-1)^3 + \frac{f^{(4)}(1)}{24}(1.01-1)^4 = -1 - 1(.01) + \frac{2}{24}(.01)^4 = -1.00999 \approx -1.01$.

6. The problem is asking you to find the second order Taylor polynomial centered at $v = 0$. Let $f(v) = e^{\frac{hv}{kT}} - 1$. Then $f'(v) = \frac{h}{kT}e^{\frac{hv}{kT}}$ and $f''(v) = \frac{h^2}{k^2T^2}e^{\frac{hv}{kT}}$. Thus $f(0) = 1 - 1 = 0$, $f'(0) = \frac{h}{kT}$, and $f''(0) = \frac{h^2}{k^2T^2}$. So $f(v) \approx \frac{hv}{kT} + \frac{h^2v^2}{2k^2T^2} = \frac{hv(2kT+hv)}{2k^2T^2}$.

7. Recall that in problem 2 (a) we determined that $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$ so the second degree approximation is $\frac{1}{(1-x)^2} \approx 1 + 2x + 3x^2$. Substituting x with $-x$ we obtain that $\frac{1}{(1+x)^2} = \frac{1}{(1-(-x))^2} \approx 1 + 2(-x) + 3(-x)^2 = 1 - 2x + 3x^2$. By taking $\frac{d}{r}$ to be x, we obtain that $E = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2} \approx \frac{kq}{r^2}\left(1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - \left(1 - 2\frac{d}{r} + 3\frac{d^2}{r^2}\right)\right) = \frac{kq}{r^2}\left(1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - 1 + 2\frac{d}{r} - 3\frac{d^2}{r^2}\right) = \frac{kq}{r^2}\left(4\frac{d}{r}\right) = \frac{4kqd}{r^3}$.