**Calculus 3**
**Lia Vas**

# Review of vectors. The dot and cross products

**Review of vectors in two and three dimensions**. A **two-dimensional vector** is an ordered pair $\vec{a} = \langle a_1, a_2 \rangle$ of real numbers. The coordinate representation of the vector $\vec{a}$ corresponds to the arrow from the origin $(0, 0)$ to the point $(a_1, a_2)$. Thus, the **length** of $\vec{a}$ is $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$. Analogously, we have the following.

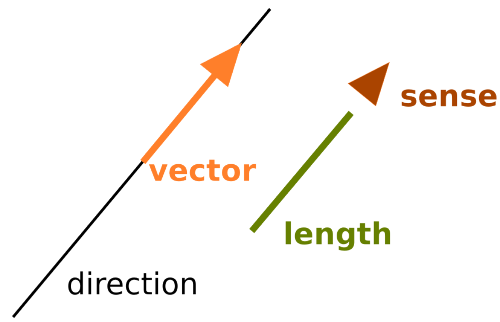
A **three-dimensional vector** is an ordered triple

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

of real numbers. The coordinate representation of the vector $\vec{a}$ corresponds to the arrow from the origin $(0, 0, 0)$ to the point $(a_1, a_2, a_3)$.

The **length** of $\vec{a}$ is

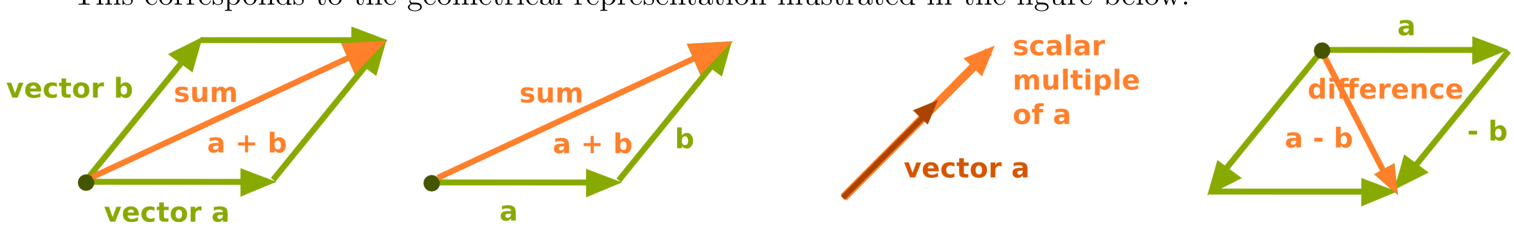
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$




Using the coordinate representation the vector addition and scalar multiplication can be realized as follows.

| | |
|---|---|
| **Vector Addition** - by coordinates | $\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$ |
| **Scalar multiplication** - by coordinates | $k\langle a_1, a_2, a_3 \rangle = \langle ka_1, ka_2, ka_3 \rangle$ |

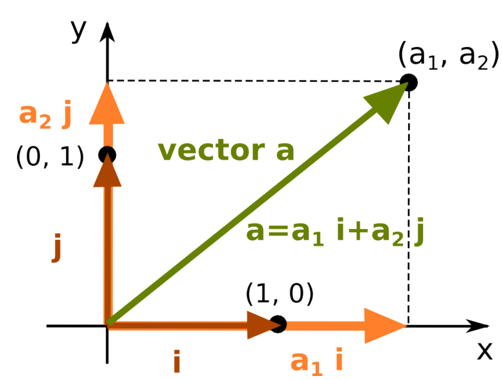
This corresponds to the geometrical representation illustrated in the figure below.




Using its coordinates, a vector $\vec{a} = \langle a_1, a_2 \rangle$ in $xy$-plane can be represented as a linear combination of vectors $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ as follows.

$$\vec{a} = a_1\vec{i} + a_2\vec{j}$$

The coordinates of a vector and geometrical representation have analogous relation in three dimensional space.



If $\vec{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \langle 0, 1, 0\rangle$, and $\vec{k} = \langle 0, 0, 1\rangle$ and a vector $\vec{a}$ can be represented as $\vec{a} = \langle a_1, a_2, a_3\rangle$, then

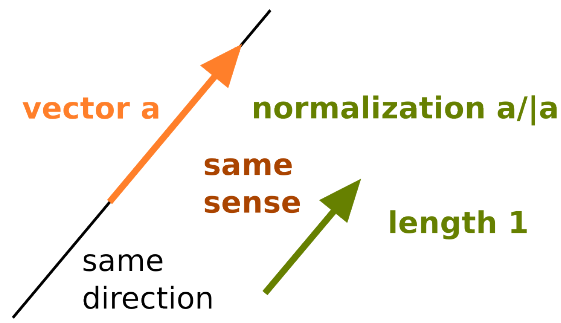
$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

In the next section, it will be relevant to determine the coordinates of the vector from one point to the other. Let $P = (a_1, a_2, a_3)$ and $Q = (b_1, b_2, b_3)$, be two points in space. If $O$ denotes the origin $(0, 0, 0)$, then the vector $\overrightarrow{OP}$ can be represented as $\langle a_1, a_2, a_3\rangle$, and the vector $\overrightarrow{OQ}$ as $\langle b_1, b_2, b_3\rangle$.

Since $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ we have that

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle b_1, b_2, b_3\rangle - \langle a_1, a_2, a_3\rangle = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3\rangle.$$

In some cases, we may need to find the vector with same direction and sense as a nonzero vector $\vec{a}$ but of length 1. Such vector is called the **normalization** of $\vec{a}$.




> The **normalization** of $\vec{a}$ is
> the vector of length 1 in the direction of $\vec{a}$,
> $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

**Practice problems.**

1. Let $P$ be the point $(2, -1)$ and $Q$ be the point $(1, 3)$. Determine and sketch the vector $\vec{PQ}$.
2. Let $\vec{a} = \langle 2, -1\rangle$ and $\vec{b} = \langle 1, 3\rangle$. Sketch $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $2\vec{a}$, $2\vec{a} - 3\vec{b}$.
3. Let $\vec{a} = \langle 3, 4, 0\rangle$ and $\vec{b} = \langle -1, 4, 2\rangle$. Determine $|\vec{a}|$, $2\vec{a} + 3\vec{b}$, $3\vec{a} - 2\vec{b}$.
4. Let $\vec{a} = \vec{i} + 4\vec{j} - 8\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 2\vec{k}$. Determine $|\vec{a}|$, $\vec{a} + \vec{b}$, $2\vec{a} - 3\vec{b}$.
5. Find the normalization of the vector $\vec{a} = \vec{i} + 4\vec{j} + 8\vec{k}$.
6. Find the normalization of the vector $\vec{a} = \langle 3, 4, 0\rangle$.

**Solutions.** 1. $\vec{PQ} = \langle -1, 4\rangle$ 3. $|\vec{a}| = 5$, $2\vec{a} + 3\vec{b} = \langle 3, 20, 6\rangle$, $3\vec{a} - 2\vec{b} = \langle 11, 4, -4\rangle$ 4. $|\vec{a}| = 9$, $\vec{a} + \vec{b} = \langle -1, 5, -6\rangle$, $2\vec{a} - 3\vec{b} = \langle 8, 5, -22\rangle$ 5. The length of $\vec{a}$ is $|\vec{a}| = \sqrt{1 + 4^2 + 8^2} = \sqrt{81} = 9$ so $\hat{a} = \frac{1}{9}\vec{i} + \frac{4}{9}\vec{j} + \frac{8}{9}\vec{k}$ 6. The length of $\vec{a}$ is $|\vec{a}| = \sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5$ so $\hat{a} = \langle \frac{3}{5}, \frac{4}{5}, 0\rangle$.

## The Dot Product

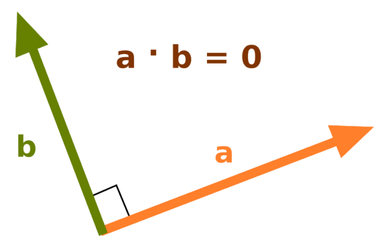
If one is to define a meaningful product of two vectors, $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, the first idea that comes to mind would probably be to consider coordinate-wise multiplication $\langle a_1b_1, a_2b_2, a_3b_3\rangle$. However, since this type of product is geometrically not very meaningful nor applicable, one consider two other types of multiplication, the dot and the cross product.

The **dot product** of vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is defined to be the scalar obtained by adding the coordinates of our first attempt to define the product, $\langle a_1 b_1, a_2 b_2, a_3 b_3 \rangle$. The notation used for such product is $\vec{a} \cdot \vec{b}$. Thus

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

This product can be used to determine the angle between the vectors and, in particular, to test whether two vectors are perpendicular to each other. If $\theta$ is the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$, then

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta}$$




As a consequence, $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal) exactly when $\cos\theta = 0$ which, in turn, happens exactly when $\vec{a} \cdot \vec{b} = 0$. Thus,

$\vec{a}$ and $\vec{b}$ are **perpendicular** if and only if $\vec{a} \cdot \vec{b} = 0$.

In case when $\vec{a} = \vec{b}$, $\theta = 0$ and $\cos\theta = 1$, and the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ becomes

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

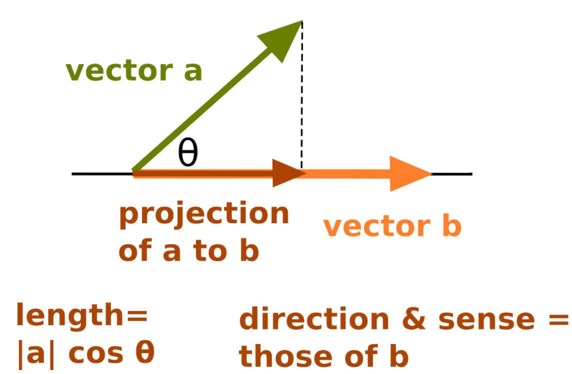
which relates the dot product and the length of a vector $\vec{a}$.

**Projection of one vector to another.** In many physics applications, it is relevant to determine the coordinates of a projection of one vector onto the other.

Let $\text{proj}_{\vec{b}}\vec{a}$ denote the **projection of $\vec{a}$ onto $\vec{b}$** for given nonzero vectors $\vec{a}$ and $\vec{b}$. The projection of $\vec{a}$ onto $\vec{b}$ has the same direction and sense as vector $\vec{b}$. The length of $\text{proj}_{\vec{b}}\vec{a}$ satisfies

$$|\text{proj}_{\vec{b}}\vec{a}| = |a| \cos\theta.$$

Thus, $\text{proj}_{\vec{b}}\vec{a}$ can be obtained by multiplying its length with the normalization of $\vec{b}$.




$$\text{proj}_{\vec{b}}\vec{a} = |\text{proj}_{\vec{b}}\vec{a}|\, \hat{b} = |\vec{a}| \cos\theta\, \hat{b} = |\vec{a}| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \frac{\vec{b}}{|\vec{b}|} \Rightarrow$$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}.$$

**Practice problems.**

1. Find the dot product of the vectors $\vec{a} = \langle 1, 3, -4 \rangle$ and $\vec{b} = \langle -2, 3, 1 \rangle$.
2. Find the angle between the vectors $\vec{a} = \langle 3, 4 \rangle$ and $\vec{b} = \langle 5, 12 \rangle$.

3. Find the angle between the vectors $\vec{a} = \langle 3, -1, 2 \rangle$ and $\vec{b} = \langle 2, 4, -1 \rangle$.

4. Find the projection of $\vec{a}$ onto $\vec{b}$ if $\vec{a} = \langle 1, -1, 0 \rangle$ and $\vec{b} = \langle 1, 0, 1 \rangle$.

**Solutions.** 1. $\vec{a} \cdot \vec{b} = -2 + 9 - 4 = 3$. 2. If $\theta$ denotes the angle between the vectors, then $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{15+48}{(5)(13)} = \frac{63}{65} = 969$. $\theta = \cos^{-1}(.969) = .249$ radians or 14.25 degrees.

3. $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{0}{|\vec{a}||\vec{b}|} = 0$. So, $\theta = 90$ degrees and the vectors are perpendicular.

4. $\mathrm{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b} = \frac{1}{(\sqrt{2})^2}\langle 1, 0, 1 \rangle = \frac{1}{2}\langle 1, 0, 1 \rangle = \langle \frac{1}{2}, 0, \frac{1}{2} \rangle$.

## The Cross Product

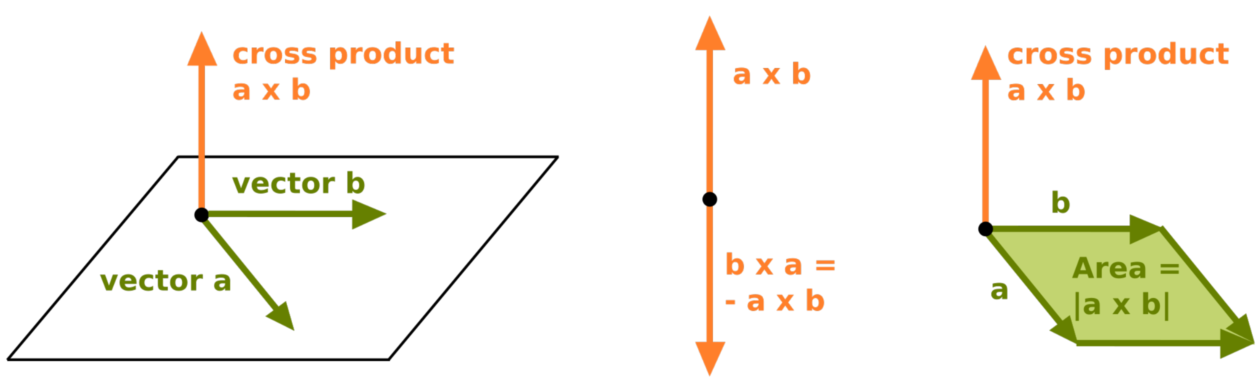
As opposed to the dot product which results in a scalar, the cross product of two vectors is again *a vector*. If $\vec{a}$ and $\vec{b}$ are two vectors, their **cross product** is denoted by $\vec{a} \times \vec{b}$.

The vector $\vec{a} \times \vec{b}$ is perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$. This determines the **direction** of $\vec{a} \times \vec{b}$. The **sense** of $\vec{a} \times \vec{b}$ is determined by the right hand rule: if $\vec{a}$ and is the thumb and $\vec{b}$ the middle finger, the index finger has the same sense as $\vec{a} \times \vec{b}$. Using the right hand rule, you can see that the cross product is not **not** commutative, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ in general, and that

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}.$$

The **length** of $\vec{a} \times \vec{b}$ is the same as the **area of the parallelogram determined by** $\vec{a}$ and $\vec{b}$. If $\theta$ is the angle between $\vec{a}$ and $\vec{b}$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$$




The cross product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ can be computed using the coordinates as follows.

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Since $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$, $\vec{a}$ and $\vec{b}$ are parallel exactly when $\sin\theta = 0$ which happens exactly when $\vec{a} \times \vec{b} = \vec{0}$. Thus,

$$\vec{a} \text{ and } \vec{b} \text{ are } \textbf{parallel} \text{ if and only if } \vec{a} \times \vec{b} = \vec{0}.$$

Another way to check if the two vectors are parallel is to check if one is a scalar multiple of the other (i.e. if $\vec{a} = k\vec{b}$ for some $k$). In this case, for $\vec{b} \neq \vec{0}$, the coordinates are such that $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

**Practice Problems.**

1. Let $\vec{a} = \langle 1, 2, 0\rangle$ and $\vec{b} = \langle 0, 3, 1\rangle$. Find $\vec{a} \times \vec{b}$.
2. Do the same for $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = \vec{j} + 2\vec{k}$.
3. Let $\vec{a} = \langle -5, 3, 7\rangle$ and $\vec{b} = \langle 6, -8, 2\rangle$. Determine if the vectors are parallel, perpendicular or neither.
4. Do the same for $\vec{a} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$.
5. Find a vector perpendicular to the plane through the points $P(1, 0, 0)$, $Q(0, 2, 0)$ and $R(0, 0, 3)$ and find the area of the triangle $PQR$.
6. Do the same for $P(0, 0, 0)$, $Q(1, -1, 1)$ and $R(4, 3, 7)$.

**Solutions.** 1. $\langle 2, -1, 3\rangle$  2. $3\vec{i} - 4\vec{j} + 2\vec{k}$

3. $\vec{a}\cdot\vec{b} = -40 \neq 0$ so the vectors are not perpendicular. Also, the coordinates are not proportional ($\frac{-5}{6} \neq \frac{3}{-8} \neq \frac{7}{2}$) so the vectors are not parallel either. Alternatively, find that the cross product is $\overrightarrow{a} \times \overrightarrow{b} = \langle 62, 52, 22\rangle \neq \langle 0, 0, 0\rangle$ so the vectors are not parallel.

4. $\vec{a} \cdot \vec{b} = 0$, thus the vectors are perpendicular.

5. Since vectors $\vec{PQ}$ and $\vec{PR}$ are in the plane, their cross product $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane. Calculate $\vec{PQ} = \langle -1, 2, 0\rangle$ and $\vec{PR} = \langle -1, 0, 3\rangle$, $\vec{PQ} \times \vec{PR} = \langle 6, 3, 2\rangle$. The area of the triangle determined by $P$, $Q$, and $R$ is half of the area of the parallelogram determined by the vectors $\vec{PQ}$ and $\vec{PR}$ which is the magnitude of $\vec{PQ} \times \vec{PR}$. Thus the triangle area is $\frac{1}{2}\sqrt{36 + 9 + 4} = \frac{7}{2}$.

6. Similarly to previous problem, find a vector perpendicular to the plane to be $\langle -10, -3, 7\rangle$ and the area of the triangle to be $\sqrt{158}/2 = 6.28$.