**Differential Equations**
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# Second and Higher Order Linear Differential Equations

A second order differential equation is **linear** if it can be written in the form

$$a(x)y'' + b(x)y' + c(x)y = g(x).$$

The general solution of such equation will depend on two constants. An **initial-value problem** for the second order equation consists of finding the solution of the second order differential equation that satisfies the conditions

$$y(x_0) = y_0 \text{ and } y'(x_0) = y_1.$$

A **boundary-value problem** for the second order equation consists of finding the solution of the second order differential equation that satisfies the conditions

$$y(x_0) = y_0 \text{ and } y(x_1) = y_1.$$

A linear differential equation is called **homogeneous** if $g(x) = 0$. To find the general solution of such differential equation, it is sufficient to find two solution $y_1(x)$ and $y_2(x)$ which are not constant multiple of one another (linearly independent solutions). Then, the general solution has the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

If a homogeneous equation has **constant coefficients** (that is if $a, b$ and $c$ are constants,) then the function of the form $y = e^{rx}$ is a solution iff $r$ is a solution of **characteristic equation**

$$ar^2 + br + c = 0$$

To see this, calculate the derivatives: $y = e^{rx}$, $y' = re^{rx}$, and $y'' = r^2 e^{rx}$ and substitute that into the equation $ay'' + by' + cy = 0$. We have that $ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx} = 0$ if and only if $r$ is a solution of $ar^2 + br + c = 0$.

Because of this, the general solution of the differential equation $ay'' + by' + cy = 0$ can be obtained by solving the characteristic equation $ar^2 + br + c = 0$. Depending on the sign of the discriminant $b^2 - 4ac$, there are three cases.

Case 1 If the characteristic equation has two real and distinct roots $r_1$ and $r_2$, then the two linearly independent solutions are $e^{r_1 x}$ and $e^{r_2 x}$ and the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

Case 2 If the characteristic equation has one real root $r$, then the two linearly independent solutions are $e^{rx}$ and $xe^{rx}$ and the general solution is $y = c_1 e^{rx} + c_2 x e^{rx}$.

Case 3 If the characteristic equation has two complex roots $p \pm iq$, then the two linearly independent solutions are $e^{px} \cos qx$ and $e^{px} \sin qx$ and the general solution is $y = c_1 e^{px} \cos qx + c_2 e^{px} \sin qx$.

**Practice Problems.**

a) Solve the following differential equations.

1. $y'' - 6y' + 8y = 0$
2. $y'' - y' - 6y = 0$
3. $y'' - 2y' + y = 0$
4. $y'' - 4y' + 4y = 0$
5. $y'' - 2y' + 2y = 0$
6. $y'' + 4y = 0$

b) Solve the following initial-value problems.

1. $y'' - 6y' + 8y = 0, \quad y(0) = 0, \quad y'(0) = 2$
2. $y'' - y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 5$
3. $y'' - 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = 3$
4. $y'' - 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 7$
5. $y'' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 2$

c) Solve the following boundary-value problems.

1. $y'' - 6y' + 8y = 0 \ \ y(0) = 0, \quad y(1) = e^2$
2. $y'' - 2y' + y = 0, \quad y(0) = 2, \quad y(1) = 0$
3. $y'' - 2y' + 2y = 0, \quad y(0) = 1, \quad y(\pi/2) = 2e^{\pi/2}$
4. $y'' + 4y = 0, \quad y(0) = 3, \quad y(\pi/4) = -2$

**Solutions.**

a)

1. $y = c_1e^{2x} + c_2e^{4x}$
2. $y = c_1e^{3x} + c_2e^{-2x}$
3. $y = c_1e^{x} + c_2xe^{x}$
4. $y = c_1e^{2x} + c_2xe^{2x}$
5. $y = c_1e^{x}\cos x + c_2e^{x}\sin x$
6. $y = c_1\cos(2x) + c_2\sin(2x)$

b)

1. $y = -e^{2x} + e^{4x}$
2. $y = e^{3x} - e^{-2x}$
3. $y = 2e^{x} + xe^{x}$
4. $y = 2e^{2x} + 3xe^{2x}$
5. $y = 2\cos(2x) + \sin(2x)$

c) 1. $y = \frac{1}{1-e^2}e^{2x} - \frac{1}{1-e^2}e^{4x}$
2. $y = 2e^x - 2xe^x$
3. $y = e^x \cos x + 2e^x \sin x$
4. $y = 3\cos(2x) - 2\sin(2x)$

## Higher Order Linear Differential Equations

The method of solving homogeneous differential equations of second order generalizes for solving homogeneous differential equations of higher order with constant coefficients.

Recall that a linear higher order differential equation is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = g(x).$$

A homogeneous linear differential equation with constant coefficients has the form

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = 0.$$

Its general solution can be determined by finding $n$ linearly independent solutions of its characteristic equation

$$a_nr^n + a_{n-1}r^{n-1} + \ldots + a_0 = 0.$$

If $r_1, r_2, \ldots, r_n$ are different real solutions, then $e^{r_1x}$, $e^{r_2x}, \ldots,$ $e^{r_nx}$ are linearly independent solutions producing a general solution

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \ldots + c_ne^{r_nx}.$$

Similarly as in the case of repeated roots, if $r_1 = r_2 = r_i$ for some $i \leq n$, then $e^{r_1x}$, $xe^{r_1x}, \ldots,$ $x^ie^{r_ix}$ are linearly independent solutions that generate a general solution.

**Examples.** Solve the equations: a) $y''' - 2y'' - y' + 2y = 0$, b) $y''' - 2y'' + y' = 0$.

**Solution.** a) The characteristic equation is $r^3 - 2r^2 - r + 2$. Factoring this equation by hand (get $(r-1)(r+1)(r-2) = 0$) or solving it using Matlab or the calculator, we obtain the solutions $r = 1, -1, 2$. Thus, the three linearly independent solutions are $e^x$, $e^{-x}$ and $e^{2x}$ and the general solution is $y = c_1e^x + c_2e^{-x} + c_3e^{2x}$.

b) The characteristic equation is $r^3 - 2r^2 + r = 0$. Factoring this equation, we get $r(r-1)^2 = 0$, so $r = 0$ is a solution and $r = 1$ is a double root. Thus, the three linearly independent solutions are $e^{0x} = 1$, $e^x$ and $xe^x$ and the general solution is $y = c_1 + c_2e^x + c_3xe^x$.

In order to better understand the complex roots case, below is a short review of complex numbers.

## Complex Numbers

Complex numbers are introduced during the course of the study of algebraic equations and, in particular, the solutions of equations that involve square roots of negative real numbers.

As usual, we denote the $\sqrt{-1}$ with $i$. A **complex number** is any expression of the form $a + ib$ where $a$ and $b$ are real numbers. $a$ is called the **real** part and $b$ is called the **imaginary** part of the complex number $a+ib$. The complex number $a-ib$ is said to be **complex conjugate** of the number $a + ib$.

**Trigonometric Representations.** Let us recall the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Using this representation, we have that

$$z = x + iy = r\cos\theta + ir\sin\theta.$$

The value $r$ is the distance from the point $(x, y)$ in the plane to the origin. The value $r$ is called the **modulus** or absolute value of $z$. The angle $\theta$ is the angle between the radius vector of $(x, y)$ and the positive part of $x$-axis. The angle $\theta$ is usually called the **argument** or **phase** of $z$.

**Euler's formula.**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This formula is especially useful in the solution of differential equations. Euler's formula was proved (in an obscured form) for the first time by Roger Cotes in 1714, then rediscovered and popularized by Euler in 1748.

For complex numbers, this allows the following simplification

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

With this formula, a lot of formulas involving complex numbers become simpler.

For example, the trigonometric representation yields an easy formula for the $n$-th power of a complex number $z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$,

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta)).$$

More importantly when solving algebraic equations of the form $z^n = a$ where $a$ is a given complex number $a = r(\cos(\theta) + i\sin(\theta))$, we can obtain $n$ solutions of the equation by the formula

$$\sqrt[n]{r}e^{\frac{(\theta+2k\pi)i}{n}} = \sqrt[n]{r}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n})$$

for $k = 0, 1, \ldots n - 1$.

**Fundamental Theorem of Algebra.** A quadratic equation $ax^2 + bx + c = 0$ can have two (possibly equal) real solutions or no real solutions. As opposed to this situation, in complex plane, every quadratic equation has exactly two solutions (possibly equal). Similar claim holds for every polynomial: Every polynomial (with complex coefficients) of degree $n$ has exactly $n$ solutions (some possibly equal). This is statement is known as the Fundamental Theorem of Algebra (more details at http://en.wikipedia.org).

Moreover, if an $n$-th degree polynomial with **real coefficient** has a complex root $a + ib$, then its complex conjugate $a - ib$ is also the root of a polynomial. Thus, *a polynomial with real coefficients of $n$-th degree has $n$ roots even number of which is complex. The complex roots appear in conjugated pairs.*

Thus, if $r_1, r_2, \ldots, r_n$ are roots of characteristic equation of a homogeneous linear differential equation with constant coefficients $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = 0$, and if $r_1 = a + ib$ and $r_2 = a - ib$ are a conjugated complex pair, then two linearly independent solutions that correspond to this conjugated pair originate from

$$e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax}(\cos bx + i \sin bx).$$

Since the solutions are real-valued functions, two solutions can be taken to be $e^{ax} \cos bx$ and $e^{ax} \sin bx$.

**Finding zeros of polynomials in Matlab.** Unlike the situation for quadratic equation, there is no general formula for polynomials of degrees higher than 4 (find out more at http://en.wikipedia.org/wiki/Polynomial#Solving_polynomial_equations). Even for cubic or quartic polynomials when such formula exists, it is rather complex to use. Thus, unless a polynomial is easy to factor or to use the $n$-th root formula, it is convenient to find approximate solutions using Matlab or some other technology.

In Matlab, you can find zeros of polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ by the command **roots**. Start by representing the polynomial as a vector of length $n+1$ with its coefficients as the entries

$$\mathbf{p{=}}[a_n \;\; a_{n-1} \;\; \ldots \;\; a_1 \;\; a_0]$$

and then use the command

**roots(p)**

**Example.** Find general solution of the equation $-90y^{(4)} + 100y^{'''} - 54y' + 16y = 0$ by using Matlab to find solutions of characteristic equation.

**Solution.** The characteristic equation is $-90r^4 + 100r^3 - 54r + 16 = 0$. Represent this polynomial in Matlab as **p=[-90 100 0 -54 16]** and use the command **roots(p)** to get the solutions $r =$ -0.6900, 0.3511 and 0.7250$\pm$0.4562$i$. This gives you the four fundamental solutions $y_1 = e^{-0.69x}$, $y_2 = e^{0.3511x}$ $y_3 = e^{0.7250x} \cos 0.4562x$ and $y_4 = e^{0.7250x} \sin 0.4562x$. So, the general solution is $y = c_1 e^{-0.69x} + c_2 e^{0.3511x} + c_3 e^{0.7250x} \cos 0.4562x + c_4 e^{0.7250x} \sin 0.4562x$.

**Practice Problems.** Find general solutions of the following differential equations.

1. $y^{(4)} - y = 0$.
2. $y^{(4)} - 5y'' - 36y = 0$.
3. $y''' - 8y = 0$
4. $y^{(5)} - 32y = 0$.
5. $y^{(5)} + 32y = 0$.
6. $-18y^{(5)} + 25y^{(4)} - 27y^{''} + 16y' + 20y = 0$. Use Matlab to find the solutions of the characteristic equation.

**Solutions.**

1. $y^{(4)} - y = 0 \Rightarrow r^4 - 1 = 0 \Rightarrow (r^2-1)(r^2+1) = 0 \Rightarrow r = \pm 1$ and $r = \pm i \Rightarrow$ the general solution is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

   Alternatively, you can find the four solutions by considering $\sqrt[4]{1} = \sqrt[4]{1e^{0i}} = 1\ e^{\frac{2k\pi}{4}i} = e^{\frac{k\pi}{2}i}$ for $k = 0,1,2,3$. Then $r_0 = 1, r_1 = i, r_2 = -1$ and $r_3 = -i$ yield the same general solution.

2. The characteristic equation $r^4 - 5r^2 - 36$ factors as $(r^2-9)(r^2+4)$. So, the solutions are $3, -3, 2i, -2i$. Thus the general solution is $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 2x + c_4 \sin 2x$.

3. $y''' - 8y = 0 \Rightarrow r^3 - 8 = 0$. You can approach this equation on two ways.

   One way is to factor $r^3 - 8$ as $(r-2)(r^2+2r+4)$ and use the quadratic formula to find zeros of the second term. Obtain $r = 2, r = -1 \pm i\sqrt{3}$. This corresponds to the solutions $y_1 = e^{2x}$, $y_2 = e^{-x}\cos\sqrt{3}x$ and $y_3 = e^{-x}\sin\sqrt{3}x$. Thus, the general solution of the differential equation is $y = c_1 e^{2x} + c_2 e^{-x}\cos\sqrt{3}x + c_3 e^{-x}\sin\sqrt{3}x$.

   Alternatively, you can find the solutions of $r^3 = 8$ by considering $\sqrt[3]{8} = \sqrt[3]{8e^{0i}} = 2\ e^{\frac{2k\pi}{3}i}$ for $k = 0,1,2$. Then $r_0 = 2e^{0i} = 2, r_1 = 2e^{\frac{2\pi i}{3}} = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -1 + \sqrt{3}i$, and $r_2 = 2e^{\frac{4\pi i}{3}} = 2(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -1 - \sqrt{3}i$. This yields the same general solution as in the previous paragraph.

4. $y^{(5)} - 32y = 0 \Rightarrow r^5 - 32 = 0 \Rightarrow r^5 = 32 = 32e^{0i} \Rightarrow \quad r_k = \sqrt[5]{32e^{0i}} = \sqrt[5]{32}e^{\frac{0+2k\pi}{5}i} = 2e^{\frac{2k\pi}{5}i}$ for $k = 0,1,\ldots,4$. $\quad r_0 = 2e^{0i} = 2$,

   $r_1 = 2e^{2\pi i/5} = 2(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}) = 0.618 + 1.902i$,

   $r_2 = 2e^{4\pi i/5} = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}) = -1.618 + 1.176i$,

   $r_3 = 2e^{6\pi i/5} = 2(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}) = -1.618 - 1.176i$,

   $r_4 = 2e^{8\pi i/5} = 2(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}) = 0.618 - 1.902i$.

   Solution 2 corresponds to $y_1 = e^{2x}$. Roots $r_1$ and $r_4$ are conjugated producing two fundamental solutions $y_2 = e^{0.618x}\cos 1.902x$ and $y_3 = e^{0.618x}\sin 1.902x$. Roots $r_2$ and $r_3$ are conjugated, producing another pair of fundamental solutions $y_4 = e^{-1.618x}\cos 1.176x$ and $y_5 = e^{-1.618x}\sin 1.176x$. Thus, the general solution is $y = c_1 e^{2x} + c_2 e^{0.618x}\cos 1.902x + c_3 e^{0.618x}\sin 1.902x + c_4 e^{-1.618x}\cos 1.176x + c_5 e^{-1.618x}\sin 1.176x$.

5. $y^{(5)} + 32y = 0 \Rightarrow r^5 + 32 = 0 \Rightarrow r^5 = -32 = 32e^{\pi i} \Rightarrow \quad r_k = \sqrt[5]{32e^{\pi i}} = \sqrt[5]{32}e^{\frac{\pi+2k\pi}{5}i} = 2e^{\frac{(2k+1)\pi}{5}i}$ for $k = 0,1,\ldots,4$. $\quad r_0 = 2e^{\pi i/5} = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}) = 1.618 + 1.176i$,

   $r_1 = 2e^{3\pi i/5} = 2(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}) = -0.618 + 1.902i, \quad r_2 = 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2$,

   $r_3 = 2e^{7\pi i/5} = 2(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}) = -0.618 - 1.902i, \quad r_4 = 2e^{9\pi i/5} = 2(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}) = 1.618 - 1.176i$. Roots $r_0$ and $r_4$ are conjugated and $r_1$ and $r_3$ are conjugated. Similarly as in the previous problem, the general solution is $y = c_1 e^{-2x} + c_2 e^{1.618x}\cos 1.176x + c_3 e^{1.618x}\sin 1.176x + c_4 e^{-0.618x}\cos 1.902x + c_5 e^{-0.618x}\sin 1.902x$.

6. The characteristic equations corresponds to a polynomial $p$ that can be represented in Matlab as **p=[-18 25 0 -27 16 20]**. The command **roots(p)** gives you the following values: 1.2971, $0.7664 \pm 0.9707i$ and $-0.7205 \pm 0.2023i$. Thus, the general solutions is $y = c_1 e^{1.2971x} + c_2 e^{0.7664x}\cos 0.9707x + c_3 e^{0.7664x}\sin 0.9707x + c_4 e^{-0.7205x}\cos 0.2023x + c_5 e^{-0.7205x}\sin 0.2023x$.