Differential Equations
Lia Vas

The Laplace Transform

The Laplace transform is an integral operator (meaning that it is defined via an integral and that it maps one function to the other). It can be useful when solving differential equations because it transforms a linear differential equation with constant coefficients into **an ordinary equation**. The Laplace transform can also be used for reduction of the order of equation of non-constant coefficients.

$$\begin{array}{ccc} \boxed{\begin{array}{c}\text{Differential Equation in } y,\\ \text{constant coefficients}\end{array}} & \xrightarrow{\text{Laplace transform}} & \boxed{\text{Ordinary Equation in } Y} \\ \uparrow & & \downarrow \\ \boxed{\text{Get solution } y.} & \xleftarrow{\text{Inverse Laplace transform}} & \boxed{\text{Find solution } Y.} \end{array}$$

The Laplace transform of a piecewise continuous and exponentially bounded function $f(t)$, defined for non-negative t-values, is defined to be

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

Note that the integral on the right side is not a function of t any more but a function of variable s. We shall denote the resulting function $\mathcal{L}[f(t)]$ by $F(s)$.

Example. Find the Laplace transform of the following functions.

a) $f(t) = 1$ b) $f(t) = e^{at}$ c) $f(t) = t$

d) $f(t) = t^2$ e) $f(t) = \sin at$ f) $f(t) = te^{at}$

Solutions. a) $\mathcal{L}[1] = \int_0^\infty e^{-st} dt = \frac{-1}{s} e^{-st}|_0^\infty = \frac{1}{s}$ for $s > 0$.
b) $\mathcal{L}[e^{at}] = \int_0^\infty e^{at-st} dt = \frac{1}{a-s} e^{at-st}|_0^\infty = \frac{-1}{a-s} = \frac{1}{s-a}$ for $s > a$.
c) $\mathcal{L}[t] = \int_0^\infty te^{-st} dt$ To evaluate this integral, use the integration by parts. We have $\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st}|_0^\infty = \frac{1}{s^2}$ for $s > 0$.
d) Using the integration by parts twice, obtain that $\mathcal{L}[t^2] = \frac{2}{s^3}$. Note that using inductive argument, it can be shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$.
e) $\mathcal{L}(\sin at) = \int_0^\infty e^{-st} \sin(at) dt$. Using the integration by parts twice, we obtain $\frac{a}{s^2+a^2}$ for $s > 0$.
f) $\mathcal{L}[te^{at}] = \int_0^\infty te^{at-st} dt$ similarly as in part c) we obtain $\frac{1}{(s-a)^2}$ for $s > a$.

The Laplace transform is linear (since the integral which defines it is linear) thus

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

for any constants a and b.

In order to transform a differential equation using the Laplace transform, we also need to know the Laplace transform of the derivatives of a function. Let Y denotes $\mathcal{L}[y]$ for some function $y(t)$. Then,

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = sY - y(0),$$
$$\mathcal{L}[y''] = s\mathcal{L}[y'] - y'(0) = s(s\mathcal{L}[y] - y(0)) - y(0) = s^2Y - sy(0) - y'(0),$$
$$\mathcal{L}[y'''] = s\mathcal{L}[y''] - y''(0) = s^3Y - s^2y(0) - sy'(0) - y''(0).$$

Continuing on this way we obtain that

$$\mathcal{L}[y^{(n)}] = s^nY - s^{n-1}y(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0).$$

The following table summarizes the Laplace transform of some frequently used functions.

Function $f(t)$	**Laplace transform** $F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
y'	$s\mathcal{L}[y] - y(0)$
$y^{(n)}$	$s^n\mathcal{L}[y] - s^{n-1}y(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)$
$u_c(t)$	$e^{-cs}\frac{1}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$\delta(t-c)$	e^{-cs}
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$

The Laplace transform of a differential equation will turn out to be an ordinary equation. However, in order to obtain a solution from the original equation from the solution of the transformed equation, we need to use the **inverse Laplace transform** $\mathcal{L}^{-1}$. Note that

$$\mathcal{L}^{-1}[F(s)] = f(t) \quad \text{if} \quad \mathcal{L}[f(t)] = F(s)$$

Thus, the inverse Laplace transform of a function in the right column of the table above is the corresponding function in the left column.

Partial Fraction Decomposition. In most cases, the function $F(s)$ will be a rational function whose partial fraction decomposition will match the functions in the above table. Thus, in order to find $\mathcal{L}^{-1}[F(s)] = f(t)$ one would need to find the partial fractions decomposition of $F(s)$. Before going to examples, we review the partial fractions decomposition from Calculus 2 course.

If $F(s) = \frac{p(s)}{q(s)}$ where p and q are polynomials such that **the degree of** p **is smaller than the degree of** q, to find the partial fraction decomposition perform the following steps.

1. Factor the denominator into a product of powers of linear terms $as + b$ and quadratic terms $as^2 + bs + c$. The quadratic equation $as^2 + bs + c = 0$ should have no real solutions otherwise you would be able to factor $as^2 + bs + c$ into a product of two linear terms.

2. For each power of a linear term of the form $(as+b)^k$, introduce k partial fractions with unknown coefficients $A_1, A_2, \ldots A_k$.

$$\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \ldots + \frac{A_k}{(as+b)^k}$$

3. For each quadratic term of the form $as^2 + bs + c$, introduce a partial fraction with unknown coefficients A and B.

$$\frac{As+B}{as^2+bs+c}$$

4. Determine the unknown coefficients by combining the partial fractions into a single fraction. Be careful when finding the least common denominator: note that it should be *equal to the initial denominator* $q(s)$. Then equate the coefficients of the numerator you obtain with the coefficients of the initial numerator $p(s)$. This should give you a system in all the unknown coefficients. Note that *the number of equations should match the number of unknowns.*

   When you determine the unknown coefficients, you have found the partial fractions decomposition.

5. Write the given rational function as a sum of partial fractions from the previous step. Each partial fraction matches some function on the right side of the table so you can find the inverse Laplace transform of each of them.

Example.

1. Find the Laplace transform of the following.

$$\text{a) } t^2e^{6t} - 7t^3 + 8, \qquad \text{b) } e^{-2t}\cos 3t + 2\sin 3t - 5.$$

2. Determine the *form* of the partial fractions of the given function (do not determine the constants).

   a) $\dfrac{2s-3}{(s^2-1)(s+2)}$ b) $\dfrac{2s-3}{(s-1)^2(s+1)}$

   c) $\dfrac{2s-3}{(s-1)^3(s+1)^2}$ d) $\dfrac{2s-3}{(s-1)(s^2+1)}$

3. Find the inverse Laplace transform of the following functions. a) $\frac{5}{s^2+4}$ b) $\frac{8}{(s-2)^4}$

   c) $\frac{10}{s^2+3s-4}$ d) $\frac{s+4}{s^2+2s+5}$ e) $\frac{5s^2+3s-2}{s^3+2s^2}$ f) $\frac{3s^2-4s+5}{(s-1)(s^2+1)}$ g) $\frac{s^2}{(s+1)^3}$

4. Solve the following initial value problems.

   a) $y'' - 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$ b) $y'' + y = \sin 2t$, $y(0) = 2$, $y'(0) = 1$.

Solution. 1. a) $\frac{2}{(s-6)^3} - \frac{42}{s^4} + \frac{8}{s}$ b) $\frac{s+2}{(s+2)^2+9} + \frac{6}{s^2+9} - \frac{5}{s}$

2. a) The denominator factors as $(s-1)(s+1)(s+2)$. The function decomposes as $\frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$.

b) The denominator factors as $(s-1)^2(s+1)$ and the function decomposes as $\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1}$.

c) There are two linear terms, the first one is with power 3 and the second with power 2. So, the function decomposes as $\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+1} + \frac{E}{(s+1)^2}$.

d) There is a linear term and a quadratic term (note that s^2+1 cannot be factored further since $s^2+1=0$ has no real solutions). So, the function decomposes as $\frac{A}{s-1} + \frac{Bs+C}{s^2+1}$.

3. a) $\mathcal{L}^{-1}[\frac{5}{s^2+4}] = \frac{5}{2}\mathcal{L}^{-1}[\frac{2}{s^2+4}] = \frac{5}{2}\sin 2t$ b) $\mathcal{L}^{-1}[\frac{8}{(s-2)^4}] = \frac{8}{6}\mathcal{L}^{-1}[\frac{3!}{(s-2)^{3+1}}] = \frac{4}{3}t^3e^{2t}$

c) Since s^2+3s-4 factors as $(s+4)(s-1)$, in order to find the Laplace transform, we need to find the partial fractions $\frac{A}{s+4} + \frac{B}{s-1}$. We obtain $A=-2$, $B=2$. So, $\mathcal{L}^{-1}[\frac{-2}{s+4} + \frac{2}{s-1}] = -2e^{-4t} + 2e^t$.

d) Since s^2+2s+5 cannot be factored in a product of two linear real terms, we have to complete to squares and to use the formulas for $e^{at}\cos bt$ and $e^{at}\sin bt$. $s^2+2s+5 = (s+2s+1)+4 = (s+1)^2+2^2$. $\frac{s+4}{s^2+2s+5} = \frac{s+1+3}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2}$. Hence the inverse Laplace is $e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$.

e) Use the partial fractions. $\frac{5s^2+3s-2}{s^3+2s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{2}{s} - \frac{1}{s^2} + \frac{3}{s+2}$. $\mathcal{L}^{-1}$ is $2 - t + 3e^{-2t}$.

f) Use the partial fractions. $\frac{3s^2-4s+5}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{2}{s-1} + \frac{s-3}{s^2+1}$ $\mathcal{L}^{-1}$ is $2e^t + \cos t - 3\sin t$.

g) $\frac{s^2}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} = \frac{A(s+1)^2+B(s+1)+C}{(s+1)^3} = \frac{As^2+2As+A+Bs+B+C}{(s+1)^3} \Rightarrow$

From the terms with s^2, $A=1$. From the terms with s, $2A+B=0$. Since $A=1$, $B=-2A=-2$. From the terms with no s, $A+B+C=0 \Rightarrow C=-B-A=2-1=1$. Thus, $\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] =$
$\mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{-2}{(s+1)^2} + \frac{1}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] = e^{-t} - 2te^{-t} + \frac{1}{2}t^2e^{-t}$.

4. a) Find the Laplace transform of the entire equation. Denoting $\mathcal{L}[y]=Y$, we get $s^2Y - 1 - 2sY + 2Y = \frac{1}{s+1}$. Solving for Y we get $Y = \frac{s+2}{(s+1)(s^2-2s+2)}$. The partial fraction decomposition is $Y = \frac{1}{5(s+1)} + \frac{1}{5}\frac{-s+8}{(s-1)^2+1} = \frac{1}{5(s+1)} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}$. From here $y = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.

b) Find the Laplace transform of the entire equation. Denoting $\mathcal{L}[y]=Y$, we get $s^2Y - 2s - 1 + Y = \frac{2}{s^2+4}$. Solving for Y we get $Y = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$. The partial fraction decomposition is $Y = \frac{2s+5/3}{s^2+1} + \frac{-2/3}{s^2+4}$. From here $y = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$.

Step, Boxcar and Impulse Functions

In the last two examples we could have used other methods to solve the differential equations (undetermined coefficients or variation of parameters) instead of the Laplace transform. The real importance of the Laplace transform is in its use for finding solutions of the equations for which the other methods fail to be applicable. In particular, Laplace transform can be used to find solutions of differential equations involving **discontinuous functions**. Most frequently used are step, boxcar and impulse functions which frequently appear in physics.

Step Function.

The **unit step or Heaviside function** is the function $u_c(t) = \begin{cases} 1, & t \geq c, \\ 0, & t < c. \end{cases}$

The Laplace transform of u_c is

$$\mathcal{L}[u_c(t)] = \int_c^\infty e^{-st}dt = \frac{-1}{s}e^{-st}|_c^\infty = \frac{e^{-cs}}{s}$$

Using the unit step function, the Laplace transform of a discontinuous function defined by

$$g(t) = \begin{cases} f(t-c), & t \geq c, \\ 0, & t < c. \end{cases} = u_c(t)f(t-c)$$

can be found to be

$$\mathcal{L}[g(t)] = \mathcal{L}[u_c(t)f(t-c)] = \int_c^\infty e^{-st} f(t-c)dt =$$

$$= \int_0^\infty e^{-s(t+c)} f(t)dt = e^{-sc} \int_0^\infty e^{-st} f(t)dt = e^{-cs}\mathcal{L}[f(t)]$$

Note that from this follows that

$$\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$$

Another property of the Laplace transform often used is that

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{-st+ct} f(t)dt = \int_0^\infty e^{-(s-c)t} f(t)dt = F(s-c)$$

Boxcar Function.

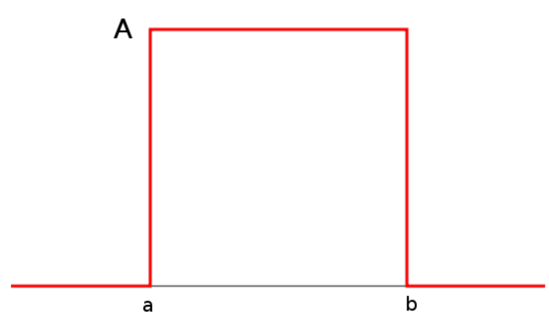
The **boxcar function** $f(t) = \begin{cases} 1, & a \leq t < b, \\ 0, & t < a \text{ and } t \geq b. \end{cases}$ can be represented as a combination of two unit step functions $u_a(t) - u_b(t)$. Indeed if $t < a < b$, both step functions are 0, so $f(t) = 0$. If $a \leq t < b$, $u_a(t) = 1$ and $u_b(t) = 0$, so $u_a(t) - u_b(t) = 1 - 0 = 1$. If $t > b$, both step functions are 1 so $u_a(t) - u_b(t) = 1 - 1 = 0$.

The Laplace transform of a boxcar function $f(t) = u_a(t) - u_b(t)$ can be found as $\mathcal{L}[u_a(t) - u_b(t)] = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$.

The **general boxcar function**

$$f(t) = \begin{cases} A, & a \leq t < b, \\ 0, & t < a \text{ and } t \geq b. \end{cases}$$

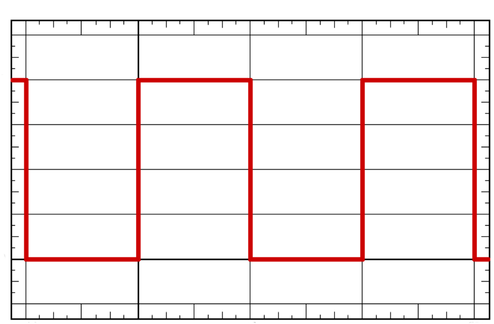
can be represented as the difference $u_a(t) - u_b(t)$ multiplied by A. So, $f(t) = A(u_a(t) - u_b(t)) = Au_a(t) - Au_b(t)$.



The **general square wave function** is a periodic extension of the boxcar function. For example, the function defined by

$$f(t) = \begin{cases} 1, & 2k \leq t < 2k+1, \\ 0, & 2k+1 \leq t < 2k+2. \end{cases}$$

for k a non-negative integer is a general square wave function. It can be represented as $f(t) = 1 - u_1(t) + u_2(t) - u_3(t) + \ldots = 1 + \sum_{k=1}^\infty (-1)^k u_k(t)$.



Example. Find the solution of differential equation

$$y'' + y = \begin{cases} 1, & 5 \leq t < 20, \\ 0, & t < 5 \text{ and } t \geq 20. \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$. Graph the solution on interval [0, 30].

Solution. The function on the right side is a boxcar function given by $u_5(t) - u_{20}(t)$. Taking the Laplace transform we obtain $s^2 Y + Y = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$. From here $Y = (e^{-5s} - e^{-20s})\frac{1}{s(s^2+1)}$. In order to find the inverse Laplace transform, it is sufficient to find the inverse Laplace transform of $\frac{1}{s(s^2+1)}$ and to use the property $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$. So let $F(s) = \frac{1}{s(s^2+1)}$ and let us find its partial fractions decomposition $F(s) = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$. Then take the Laplace inverse $\mathcal{L}^{-1}[F(s)] = 1 - \cos t$. Denote this last expression by $f(t)$. Thus, the solution is

$$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[(e^{-5s}F(s) - e^{-20s}F(s)] = u_5(t)f(t-5) - u_{20}(t)f(t-20) =$$

$$= u_5(t)(1 - \cos(t-5)) - u_{20}(t)(1 - \cos(t-20)).$$

In order to graph the solution using Matlab or a graphing calculator, we can represent it using curly bracket notation. The resulting function will consist of three branches.

1. Since both $u_5(t)$ and $u_{20}(t)$ are zero for $t < 5$, we obtain the first branch of the function to be zero for $t < 5$.

2. On the interval $5 \leq t < 20$, the function $u_5(t)$ is 1 and $u_{20}(t)$ is 0. Thus, the second branch is $1(1 - \cos(t-5)) - 0(1 - \cos(t-20)) = 1 - \cos(t-5)$ for $5 \leq t < 20$.

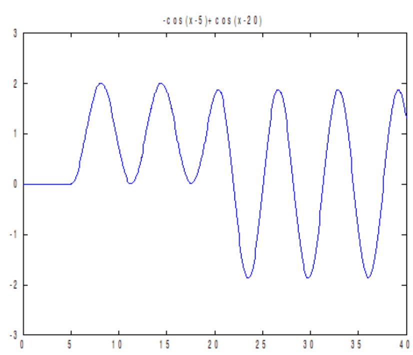
3. On the interval $t \geq 20$, both functions $u_5(t)$ and $u_{20}(t)$ are 1. Thus, the third branch is $1(1 - \cos(t-5)) - 1(1 - \cos(t-20)) = -\cos(t-5) + \cos(t-20)$ on $t \geq 20$.

Hence

$$y = \begin{cases} 0, & t < 5 \\ 1 - \cos(t-5), & 5 \leq t < 20, \\ -\cos(t-5) + \cos(t-20), & t \geq 20. \end{cases}$$

To graph y using Matlab, you can use

syms x
ezplot(0*x, [0, 5])
hold on
ezplot(1-cos(x-5), [5, 20])
ezplot(-cos(x-5)+cos(x-20), [20, 40])
hold off
axis([0 40 -3 3])



To graph y using TI83+, you can enter

0(X<5)+(1-cos(X-5))(5≤ X<20)+(-cos(X-5)+cos(X-20))(X≥ 20)

as a function and graph it. The inequality signs <, ≥ and other can be found in **2nd Math** menu.

Impulse function.

In many applications, it is necessary to represent phenomena of an impulsive nature using mathematical models. For example, voltages of large magnitude that act over a very short period of time. Consider a function δ_h defined by

$$\delta_h(t) = \begin{cases} \frac{1}{2h}, & -h < t < h, \\ 0, & t \le -h \text{ and } t \ge h. \end{cases}$$

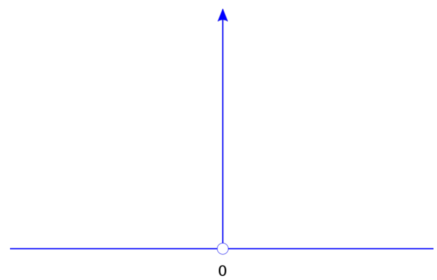
Note that

$$\int_{-\infty}^{\infty} \delta_h(t)dt = \int_{-h}^{h} \frac{1}{2h}dt = \frac{1}{2h}t|_{-h}^{h} = \frac{1}{2h}2h = 1$$

Since this integral does not depend on h, $\int_{-\infty}^{\infty} \delta_h(t)dt = 1$ even if we let $h \to 0$.

Now let us define

$$\delta(t) = \lim_{h \to 0} \delta_h(t).$$



The impulse function $\delta(t)$ is characterized by the following properties

1. $\delta(t) = 0$ for all values of $t \neq 0$
2. $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

Since no ordinary function satisfies both of these properties, δ is not a function in the usual sense of the word. It is an example of a generalized function. $\delta(t)$ is called **unit impulse function** or **Dirac delta function**. Alternate definition of Dirac delta function can be found on wikipedia.

The Laplace transform of $\delta(t)$ can be found to be 1 (to see this, find the Laplace transform of $\delta_h(t)$ and let $h \to 0$). More generally,

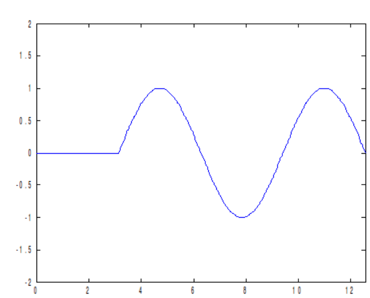
$$\mathcal{L}[\delta(t-c)] = e^{-cs}$$

Example. Find the solution of differential equation $y'' + y = \delta(t - \pi)$ with the initial conditions $y(0) = 0$ and $y'(0) = 0$. Sketch the graph of the solution.

Solution. Taking the Laplace transform, we obtain $s^2Y + Y = e^{-\pi s}$. Thus $Y = e^{-\pi s}\frac{1}{s^2+1}$ and so $y = u_\pi(t)\sin(t - \pi)$.

To graph the solution, represent it as

$$y = \begin{cases} 0, & t < \pi \\ \sin(t - \pi), & t \ge \pi. \end{cases}$$



Use **ezplot** in combination with **hold on** and **hold off** to graph it in Matlab. On TI83, you can use **0(X$<\pi$)+sin(X-π)(X$\geq\pi$)**.

Integral Equations. The Convolution

The convolution is an operation on two functions $f(t)$ and $g(t)$ defined as follows

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

The significance of this operation for the Laplace transform is because

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$$

This property implies that $\mathcal{L}^{-1}[F(s) \cdot G(s)] = \mathcal{L}^{-1}[F(s)] * \mathcal{L}^{-1}[G(s)]$.
Using convolution, the function equations of involving or yielding an integral could also be solved.

Example. Solve the integral equation $y(t) + \int_0^t (t-\tau)y(\tau)d\tau = 1$.
Solution. Note that the equation can be written as $y + t * y = 1$. Let again $Y = \mathcal{L}[y]$. Taking the Laplace transform, we have that $Y + \frac{1}{s^2}Y = \frac{1}{s}$. Solving for Y, we have $Y = \frac{s}{s^2+1}$. Thus, $y = \cos t$.

The equation from the last example belongs to the class of integral equations known as Volterra integral equations. This class of equations was introduced in the early 1900s, by V. Volterra in his study of population growth. The general form of these equations is $y(t) + \int_0^t f(t-\tau)y(\tau)d\tau = g(t)$.

Using convolution, one can also express solution of differential equation involving *any* integrable function $f(t)$ in terms of an integral involving $f(t)$.
Example. Find the solution of the equation $y'' + y = f(t)$, with the initial conditions $y(0) = 0$ and $y'(0) = 0$. Express your answer in terms of an integral involving function $f(t)$.
Solution. Let $Y = \mathcal{L}[y]$ and $F = \mathcal{L}[f]$. The Laplace transform converts the equation to $s^2Y + Y = F$. From here $Y = \frac{F}{s^2+1} = F\frac{1}{s^2+1}$. Applying the inverse Laplace, you obtain

$$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[F\frac{1}{s^2+1}] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[\frac{1}{s^2+1}] = f * \sin t = \int_0^t f(\tau)\sin(t-\tau)d\tau.$$

Solving Systems Using Laplace Transform

The Laplace transform can be used for solving systems of differential equations. Taking Laplace transform of a system of differential equations produces a system of *ordinary* equations. Solving this new system and taking the inverse Laplace transform of the solution produces the solutions of the original system.
Example. Solve the following system.

$$\begin{aligned} x' &= -x - 3y \qquad & x(0) = 1 \\ y' &= -x + y \qquad & y(0) = 0 \end{aligned}$$

Solutions. Let X denote $\mathcal{L}[x]$ and Y denote $\mathcal{L}[y]$. Start by taking the Laplace transform of both equations. Obtain the new system

$$sX - 1 = -X - 3Y \qquad sY = -X + Y$$

Solving the second equation for X produces $X = Y - sY$. Substitute that in the first equation and solve for Y. $s(Y - sY) - 1 = -Y + sY - 3Y \Rightarrow sY - s^2Y - 1 = sY - 4Y \Rightarrow -1 = (s^2-4)Y \Rightarrow Y = \frac{-1}{s^2-4}$.

Thus $X = Y - sY = \frac{-1}{s^2-4} + \frac{s}{s^2-4}$. Finding the partial fractions decomposition for X and Y produces $Y = \frac{-1/4}{s-2} + \frac{1/4}{s+2}$ and $X = \frac{1/4}{s-2} + \frac{3/4}{s+2}$. Thus, $x = \mathcal{L}^{-1}[X] = \frac{1}{4}e^{2t} + \frac{3}{4}e^{-2t}$ and $y = \mathcal{L}^{-1}[Y] = \frac{-1}{4}e^{2t} + \frac{1}{4}e^{-2t}$.

Practice Problems.

1. Find the Laplace transform of the following functions.

   a) $t^4 e^{-2t} + \cos 5t - 7$ b) $\int_0^t \tau^3 e^{t-\tau} d\tau$ c) $\int_0^t \sin(2\tau)\cos(2t - 2\tau) d\tau$

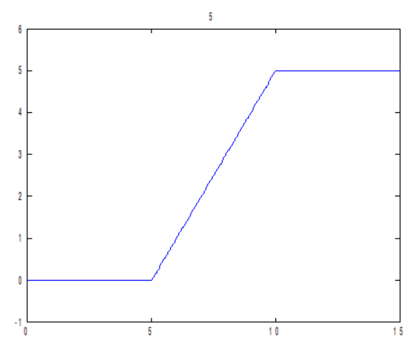
2. Find the inverse Laplace transform of the following functions.

   a) $\frac{3}{s^2+25}$ b) $\frac{5}{(s+2)^3}$ c) $\frac{s-11}{s^2+2s-3}$ d) $\frac{s-2}{s^2-2s+5}$ e) $\frac{7s^3-2s^2-3s+6}{s^4-2s^3}$ f) $\frac{7s^2-41s+84}{(s-1)(s^2-4s+13)}$

3. Use the Laplace transform to solve the following initial value problems.
   a) $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.
   b) $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$.
   c) $y'' - 6y' + 5y = 2$, $y(0) = 0$, $y'(0) = -1$.

4. Solve the initial value problem $y'' + 4y = \begin{cases} 0, & t < 5, \\ t-5, & 5 \leq t < 10, \\ 5, & t \geq 10. \end{cases}$ with $y(0) = 0$ and $y'(0) = 0$.

   The function on the right side of the equation is known as **ramp loading** and can be represented as $u_5(t)(t-5) - u_{10}(t)(t-10)$. Assume that an undamped harmonic oscillator is described by the given differential equation. Consider the graph of the motion and expand on its meaning.



5. Solve the initial value problem $y'' + 4y = \delta(t - 4\pi)$, $y(0) = 1$, $y'(0) = 0$.

6. Assume that the initial value problem $y'' + 3y' + 4y = \delta(t-3)$, $y(0) = 0$, $y'(0) = 0$ models the motion y (in cm) of an oscillator as time t (in seconds) passes. Find the solution, write your answer as a piecewise function, sketch its graph and describe the motion of the oscillator.

7. Solve the initial value problem $y'' + 4y = f(t)$, $y(0) = 3$, $y'(0) = -1$. Express your answer in terms of an integral involving function $f(t)$.

8. Solve the integral equation $y(t) + \int_0^t (t-\tau)y(\tau)d\tau = t$.

9. Solve the integro-differential equation $y'(t) + \int_0^t y(t-\tau)e^{-2\tau}d\tau = 1$, $y(0) = 1$.

10. Solve the following system.

$$\begin{aligned} x' &= -x + y \qquad & x(0) = 1 \\ y' &= -x - y \qquad & y(0) = 2 \end{aligned}$$

Solutions.

1. a) $\frac{24}{(s+2)^5} + \frac{s}{s^2+25} - \frac{7}{s}$ b) The function is the convolution of t^3 and e^t. Thus the Laplace transform is $\mathcal{L}[t^3]\mathcal{L}[e^t] = \frac{6}{s^4}\frac{1}{s-1} = \frac{6}{s^4(s-1)}$. c) The function is the convolution of $\sin 2t$ and $\cos 2t$. Thus the Laplace transform is $\mathcal{L}[\sin 2t]\mathcal{L}[\cos 2t] = \frac{2}{s^2+4}\frac{s}{s^2+4} = \frac{2s}{(s^2+4)^2}$.

2. a) $\frac{3}{5}\sin 5t$ b) $\frac{5}{2}t^2e^{-2t}$ c) $3e^t - 2e^{-3t}$ d) $e^t\cos 2t - \frac{1}{2}e^t \sin 2t$ e) $1 - \frac{3}{2}t^2 + 6e^{2t}$ f) $5e^t + 2e^{2t}\cos 3t - 5e^{2t}\sin 3t$

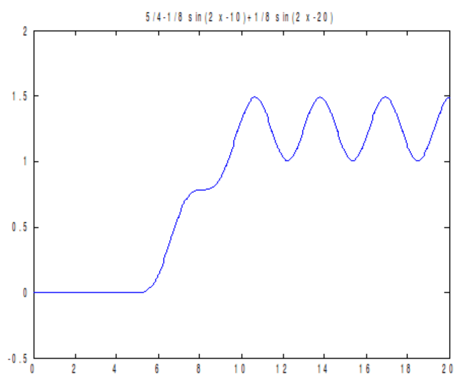
3. a) The Laplace transform of the equation is $s^2Y - s + 3sY - 3 + 2Y = 0 \Rightarrow Y = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$. The partial fraction decomposition is $Y = \frac{2}{s+1} - \frac{1}{s+2}$. Thus $y = 2e^{-t} - e^{-2t}$.

   b) The Laplace transform of the equation is $s^2Y - s - 1 - 4sY + 4 + 4Y = 0 \Rightarrow Y = \frac{s-3}{s^2-4s+4} = \frac{s-3}{(s-2)^2}$. The partial fraction decomposition is $Y = \frac{1}{s-2} - \frac{1}{(s-2)^2}$. Thus $y = e^{2t} - te^{2t}$.

   c) The Laplace transform of the equation is $s^2Y + 1 - 6sY + 5Y = \frac{2}{s} \Rightarrow Y(s^2 - 6s + 5) = \frac{2}{s} - 1 \Rightarrow Y(s-1)(s-5) = \frac{2-s}{s} \Rightarrow Y = \frac{2-s}{s(s-5)(s-1)}$. The partial fraction decomposition is $Y = \frac{2}{5s} - \frac{3}{20(s-5)} - \frac{1}{4(s-1)}$. Thus $y = \frac{2}{5} - \frac{3}{20}e^{5t} - \frac{1}{4}e^t$.

4. Note that the function on the right side of the equation can be represented as $u_5(t)(t-5) - u_{10}(t)(t-10)$. The Laplace transform makes the equation into $s^2Y + 4Y = e^{-5s}\frac{1}{s^2} - e^{-10s}\frac{1}{s^2}$. Thus $Y = (e^{-5s} - e^{-10s})\frac{1}{s^2(s^2+4)}$. The fraction $\frac{1}{s^2(s^2+4)}$ decomposes as $\frac{1/4}{s^2} - \frac{1/4}{s^2+4}$ and its inverse Laplace transform is $f(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$. Thus,

   $y = \mathcal{L}^{-1}[Y] = u_5(t)f(t-5) - u_{10}(t)f(t-10) = u_5(t)(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)) - u_{10}(t)(\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)) =$
   $$= \begin{cases} 0, & 0 \le t < 5, \\ \frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5), & 5 \le t < 10, \\ \frac{5}{4} - \frac{1}{8}\sin 2(t-5) + \frac{1}{8}\sin 2(t-10), & t \ge 10. \end{cases}$$
   The solution is zero before 5. Between 5 and 10, it increases by oscillating about the line $\frac{1}{4}(t-5)$. For $t \ge 10$, the graph becomes one of a simple harmonic oscillation (like shifted sine function) oscillating about 5/4.



5. The Laplace transform gives you $s^2Y - s + 4Y = e^{-4\pi s}$. Thus $Y = \frac{e^{-4\pi}+s}{s^2+4}$. Then $y = u_{4\pi}(t)\frac{1}{2}\sin 2(t-4\pi) + \cos 2t = \begin{cases} \cos 2t, & t < 4\pi, \\ \cos 2t + \frac{1}{2}\sin 2(t-4\pi) & t \ge 4\pi. \end{cases}$

6. Let $Y = \mathcal{L}[y]$. Applying the Laplace transform to the equation $y'' + 3y' + 4y = \delta(t-3)$ with $y(0) = y'(0) = 0$ produces $s^2Y + 3sY + 4Y = e^{-3s}$. From here $Y = \frac{e^{-3s}}{s^2+3s+4}$. Complete the denominator of $F(s) = \frac{1}{s^2+3s+4}$ to a sum of squares. $s^2 + 3s + 4 = s^2 + 2s(\frac{3}{2}) + \frac{9}{4} + 4 - \frac{9}{4} = (s+\frac{3}{2})^2 + \frac{7}{4}$. Thus $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{1}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}\mathcal{L}^{-1}[\frac{\frac{\sqrt{7}}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}e^{-3t/2}\sin\frac{\sqrt{7}}{2}t$.

   $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{e^{-3s}}{s^2+3s+4}] = \mathcal{L}^{-1}[e^{-3s}F(s)] = u_3(t)\,f(t-3) = u_3(t)\,\frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3) \Rightarrow$

$$y = \begin{cases} 0, & t < 3, \\ \frac{2}{\sqrt{7}} e^{-3(t-3)/2} \sin \frac{\sqrt{7}}{2}(t-3), & t \geq 3. \end{cases}$$

The object starts oscillating after the first three second. It oscillates with a decreasing amplitude given by $\frac{2}{\sqrt{7}} e^{-3(t-3)/2}$ converging to 0. So the oscillations become negligible in time.

7. Let $Y = \mathcal{L}[y]$ and $F = \mathcal{L}[f]$. The Laplace transform converts the equation to $s^2Y - 3s + 1 + 4Y = F \Rightarrow Y(s^2+4) = 3s - 1 + F \Rightarrow Y = \frac{3s-1}{s^2+4} + F\frac{1}{s^2+4} = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}F\frac{2}{s^2+4}$. Applying the inverse Laplace transform, you obtain $y = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}f(t) * \sin 2t = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t f(\tau)\sin 2(t-\tau)d\tau$.

8. The equation is $y + t * y = t$. The Laplace transform gives you $Y + \frac{1}{s^2}Y = \frac{1}{s^2}$. Then $Y = \frac{1}{s^2+1}$ and so $y = \sin t$.

9. The equation is $y' + y * e^{-2t} = 1$. Thus $sY - 1 + Y\frac{1}{s+2} = \frac{1}{s} \Rightarrow Y(s + \frac{1}{s+2}) = \frac{1}{s} + 1 \Rightarrow Y\frac{s(s+2)+1}{s+2} = \frac{1+s}{s} \Rightarrow Y = \frac{(1+s)(s+2)}{s(s^2+2s+1)} = \frac{(1+s)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)}$. Find the partial fraction decomposition to be $Y = \frac{2}{s} - \frac{1}{s+1} \Rightarrow y = 2 - e^{-t}$.

10. Let $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$. Taking $\mathcal{L}$ of both equations gives you $sX - 1 = -X + Y$ and $sY - 2 = -X - Y$. From the first equation $Y = sX + X - 1$. Plugging that in the second gives you $s(sX + X - 1) - 2 = -X - (sX + X - 1) \Rightarrow s^2X + 2sX + 2X = s + 3 \Rightarrow X = \frac{s+3}{s^2+2s+2}$. Thus $Y = \frac{s^2+3s+s+3-s^2-2s-2}{s^2+2s+2} = \frac{2s+1}{s^2+2s+2}$.
    Then $x = \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}[\frac{s+1+2}{(s+1)^2+1}] = \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1}] = e^{-t}\cos t + 2e^{-t}\sin t$ and $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{2s+1}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{2s+2-1}{(s+1)^2+1}] = \mathcal{L}^{-1}[2\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}] = 2e^{-t}\cos t - e^{-t}\sin t$.