Matlab Notes for Differential Equations

Lia Vas

Content

1. M-files

Suppose that you want to perform the same operation many times for different input values. Matlab allows you to create a function or a script that you can execute repeatedly with different input values (i.e. to do programming). Such function or a script is called an M-file. An M-file is an ordinary text file containing Matlab commands. You can create them using any text editor that can save files as plain (ASCII) text. You can also create an M-file by using "File" menu and choosing "New M-file" option from your Matlab command window. An important fact to keep in mind when using M-files is that an M-file can be executed from the command window just if the "Current Directory" on the top of the command window is set to be **the same directory** where the M-file is saved.

Example 1.1. Suppose that you want to solve the equation $ax^2 + bx + c = 0$ using the quadratic formula. We can create a function which will have as input values of *a, b* and *c* and which will give the solution(s) of the quadratic equation. We can enter the following in an M-file.

```
function [x1 x2] = quadratic1(a, b, c)
x1 = (-b+sqrt(b^2-4*a*c))/(2*a);
x2 = (-b-sqrt(b^2-4*a*c))/(2*a);
```

To execute the M-file and find solutions of $x^2 - 3x + 2 = 0$, you can use

```
>> [x1, x2] = quadratic(1, -3, 2)
```

The output is **x1 = 2** **x2 = 1**

Note that the semi-colon symbol after the commands suppresses display of the results of these commands. The values of x1 and x2 will still be displayed since we listed these variables in the brackets of the function.

Alternatively, the following M-file can be used.

```
function [ ] = quadratic2(a, b, c)
```

```
x1 = (-b+sqrt(b^2-4*a*c))/(2*a)
x2 = (-b-sqrt(b^2-4*a*c))/(2*a)
```

Here we do not need to list the variables in the first line since we did not use the semi-colon symbols when computing the values of x1 and x2. Also, you do not need to list **x1** and **x2** in the first line since they will be displayed when 2nd and 3rd lines are executed. Also, you can execute this M-file to solve $x^2 -3x + 2 = 0$ simply as **quadratic(1, -3, 2)** instead of **[x1, x2] = quadratic(1, -3, 2)** The output is the same as for the first version.

```
>> quadratic(1, -3, 2)
```

This means that you want to solve the equation. Matlab gives you the answers:

```
x1 =     2                              x2 =     1
```

These M-files produce the complex values if the solutions of a quadratic equation are complex numbers. For example, to solve $x^2 + 4 = 0$, we can use

```
>> quadratic(1, 0, 4)
x1 =       0 + 2.0000i                  x2 =       0 - 2.0000i
```

<u>**Example 1.2**</u>. Suppose that we need to write a program that calculates the Cartesian coordinates *(x,y)* of a point given by polar coordinates *(r,θ)*.
One possible solution is the following M-file.

```
function [x, y] = polar(r, theta)
x = r*cos(theta);
y = r*sin(theta);
```

For example, to calculate *(x,y)* coordinates of for r=3 and $\theta = \pi$, we can use the command **[x,y]=polar(3, pi)** and get the answer

```
x = -3                    y =  3.6739e-016
```

Note that y is very close to 0. The inaccuracy comes from the fact that using **pi** for π in Matlab gives an approximation of π not the exact value. The exact representation of π is obtained by **sym('pi')**. The command **[x,y]=polar(3, sym('pi'))** gives you the expected answer

```
x = -3                    y = 0.
```

2. Basics of First Order Differential Equations

To find symbolic solution of a differential equation, you can use the command **dsolve**. To use **dsolve**, the derivative of the function y is represented by **Dy**. The command has the following form:

dsolve('*equation*', '*independent variable*')

If we have the initial condition, we can get the particular solution on the following way:

dsolve('*equation*', '*initial condition*', '*independent variable*')

<u>**Example 2.1**</u>. Consider the equation $x\, y' - y = 1$. a) Find the general solution. b) Find the particular solution corresponding to the initial condition $y(1)=5$ and graph it.

a) You can find the general solution by using: **>> dsolve('x*Dy-y=1', 'x')**
You obtain **ans = -1+x*C1** This means that the solution is any function of the form y = -1+

cx, where *c* is any constant.

b) To find the solution of $x\,y' - y = 1$ with the initial condition $y(1)=5$, we can use:
>> dsolve('x*Dy-y=1', 'y(1)=5', 'x') **ans = -1+6*x**
To graph this solution, we can use: **>> ezplot(ans)**

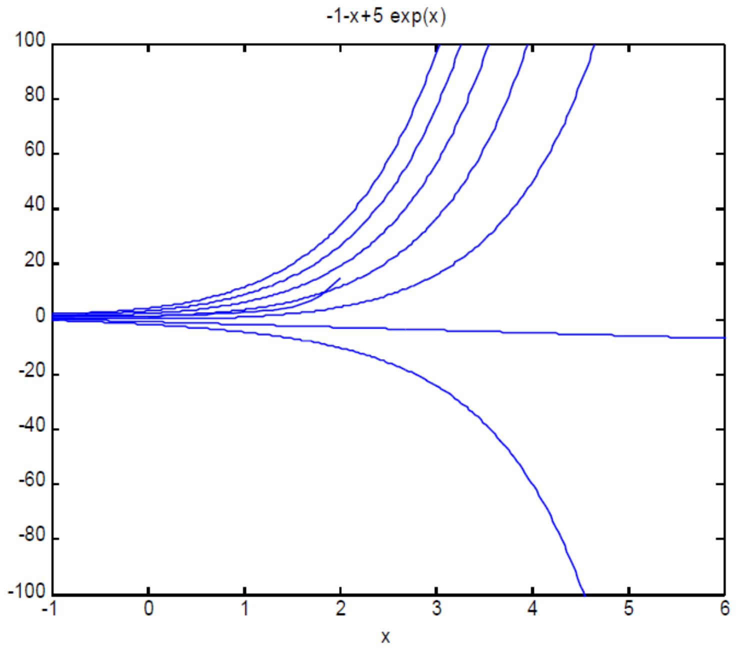
We can graph a couple of different solutions on the same chart.

<u>Example 2.2.</u> Graph the solutions of the equation $y' = x+y$ for $y(0)$ taking integer values between -2 and 4.

First we can find the solution with the initial condition of the form $y(0)=c$.
>> s = dsolve('Dy = x+y', 'y(0)=c', 'x')
s = -x-1+exp(x)*(1+c)

Then, we can use the following M-file to produce the graph of the required particular solutions.

```
function[ ]=many_solutions(s)
for cval=-2:1:4               (this means that the variable cval is taking values starting at -2 and
                              ending at 4 with step size of 1. Change this line if you need to use
                              different values of c)
   hold on
   ezplot(subs(s, 'c', cval)) (this means that the value of c in s is substituted by values of
                              cval. Add the domain of the solutions to this command as
                              explained in Review of Matlab if necessary)
   hold off
end
```



You can execute the M-file by
>> many_solutions(s)

You can modify the window by using the command
axis([$x_{min},\ x_{max},\ y_{min},\ y_{max}$**])**

For example, in the previous example you can use **axis([-1 6, -100 100]).**

By changing **s** and values of **c**, you can modify the M-file to produce graphs of solutions of different equations as the next example illustrates.

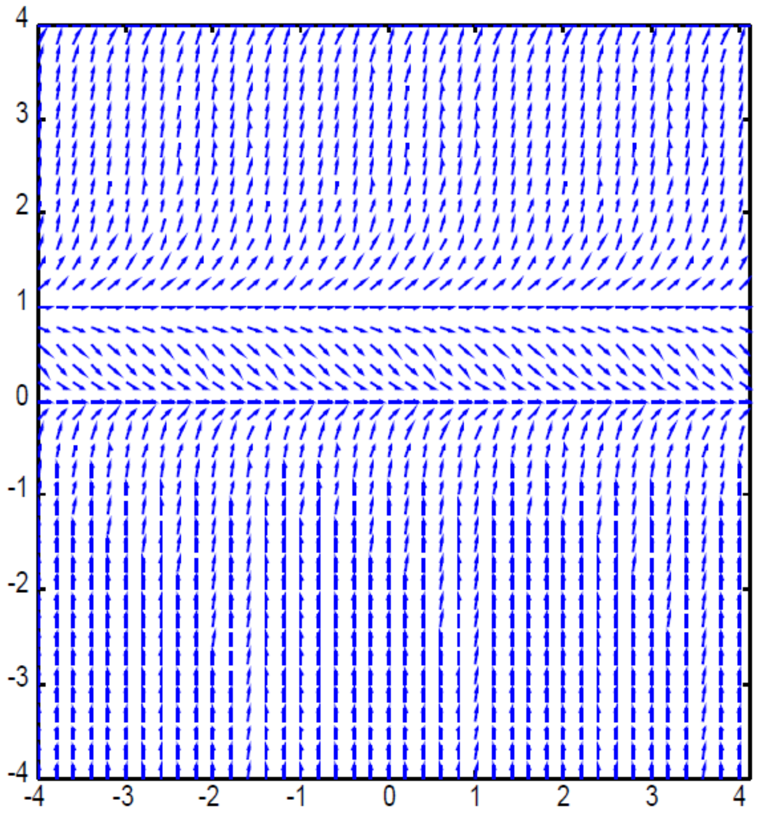
<u>Example 2.3.</u> Let us graph the solutions of the equation $y' = (y-2)(100-y)$ for sufficiently many values of $y(0)$ so that the limiting behavior of all solutions can be determined.

First we solve the equation naming the general solution s.

>> s = dsolve('Dy = (y-2)*(100-y)', 'y(0)=c', 'x')

Since this is an autonomous equation with the equilibrium solutions at y=2 and y=100, in order to get a telling graph, we can graph a few solutions with initial conditions less than 2, a few between 2 and 100 and a few above 100. We can use the same M-file as in the previous example, but with a few modification. For one, we need to modify the line that specifies the values of initial condition c. For example we can take the values of c to be 0, 10, 20, ... , 150. So, we start at 0 and end with 150 making the step size of 10. In Matlab, we can represent this by 0:10:150. Thus, substitute the line **cval=-2:1:4** with **cval=0:10:150** and use the M-file **many_solutions** again. Secondly, to see the graphs for t values larger than 6, [0, 15] for example, modify the line with ezplot to be **ezplot(subs(s, 'c', cval), [0 15])**. Finally, a nice window for this graph can be **axis([0 15 0 150])** for example.

3. Direction Fields



Suppose that we need to plot the direction field for a first order linear equation. We can do that by using the command **quiver** together with the command **meshgrid**. The following M-file displays the direction field of the first order differential equation $y' = f(x,y)$ for x and y values between -4 and 4.

function[ ]=direction_field(f)
[x,y]=meshgrid(-4:0.2:4, -4:0.2:4); (by changing this line you can change the x and y values)
S=f(x,y);
K=inline(vectorize('1/sqrt(1+S*S)'), 'S');
L=K(S);
quiver(x,y,L,S.*L,0.5)
axis equal tight

__**Example 3.1.**__ Graph the direction field of the autonomous differential equation $y' = y(4-y)(y-1)$.

Inline the function on the right side first

>> f=inline(vectorize('y*(4-y)*(y-1)'), 'x', 'y');

and then execute the M-file

>> direction_field(f)

The outcome is the graph above. From the graph, we can check the stability of equilibrium solutions y=0, y=1 and y=4.

4. Numerical solutions using **ode45**

Many differential equations cannot be solved explicitly in terms of elementary functions. for those equations, approximate solutions can be obtained using numerical methods. Approximate solutions can be found by using the command **ode45**. We can illustrate the use of this command on the following example.

Example 4.1. Consider the initial value problem y' = e^{-x^2} , y(0)=1. Graph the solution on the interval [0, 2] and display the y-values for x=0, 0.5, 1, 1.5 and 2.

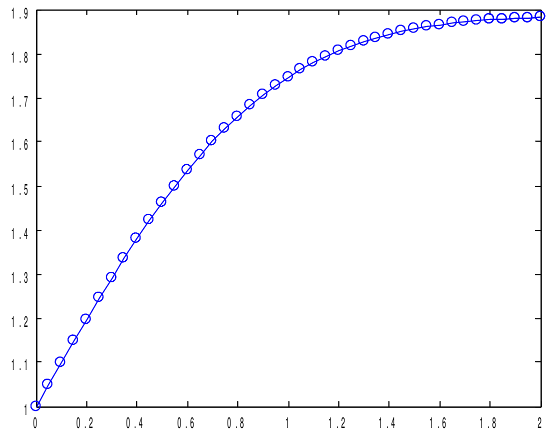
The command **ode45** requires that the equation is in the form *y'=f(x,y)*. Thus, if the equation is not given in this form, you have to solve for *y*' first. Then you want to represent the function on the right side of the equation as a vector function. You can use the command:

```
>> f=inline(vectorize('exp(-x^2)'),'x','y');
```

You can use the command **ode45** to plot the graph of the initial value problem on the specified interval. In this example, we can use

```
>> ode45(f, [0 2], 1)
```

to graph the solution of y'=f(x,y) on the interval [0, 2] for initial condition y(0)=1.



We can obtain the numerical values of the solution as well by using:

```
>> [x, y] = ode45(f, [0 2], 1)
```

The values between 0 and 2 at which **ode45** calculates the solution are stored in *x*, and the value of solution at these values is stored in *y*.
If you want the x and y values to be displayed in two columns next to each other (so that it is easy to see the y-value for a corresponding x-value), simply type

```
>> [x,y]
```

If you want the graph without the circles around the points at which **ode45** calculates the solution, you can do the following:

```
>> [x, y] = ode45(f, [0 2], 1);
>> plot(x, y)
```

If we do not want to display the entire list of (x,y)-values but just the y-values for x=0, 0.5, 1, 1.5 and 2, we can use the command:

```
>> [x,y]=ode45(f, [0:.5:2], 1);
>> [x,y]
```

The output will be:

```
x =                y =
     0             1.0000
  0.5000           1.4613
  1.0000           1.7468
```

```
   1.5000          1.8562
   2.0000          1.8821
```

5. Euler Method

The following M-file calculates the numerical values of the solution of an initial value problem using the Euler method. It approximates the solution of the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ on the interval $[x_0, x_n]$ using n steps using the following formulas:

$$h = (x_n - x_0)/n, \qquad x_{i+1} = x_i + h, \quad \text{and} \quad y_{i+1} = y_i + f(x_i, y_i)\, h$$

for all i=0,1...,n-1. The input is the **inline** function f, x_0, y_0, x_n and n. The output is the list of x and y values of the approximate solution.

```
function [x, y] = euler(f, xinit, yinit, xfinal, n)
h = (xfinal - xinit)/n;  (calculates the step size)
x = zeros(n+1, 1);
y = zeros(n+1, 1);  (initialize x and y as column vectors of size n+1)
x(1) = xinit;
y(1) = yinit; (the first entry in the vectors x and y is x0 and y0 respectively)
for i = 1:n
  x(i + 1) = x(i) + h; (every entry in vector x is the previous entry plus the step size h)
  y(i + 1) = y(i) + h*f(x(i), y(i)); (Euler Method formula)
end
```

Example 5.1. Approximate the y-value of the solution of the initial value problem y' = e^{-x^2}, y(0)=1 for x=2. Use Euler method with 20 steps. Display the y-values for x=1 and x=2 and the graph of the solution.

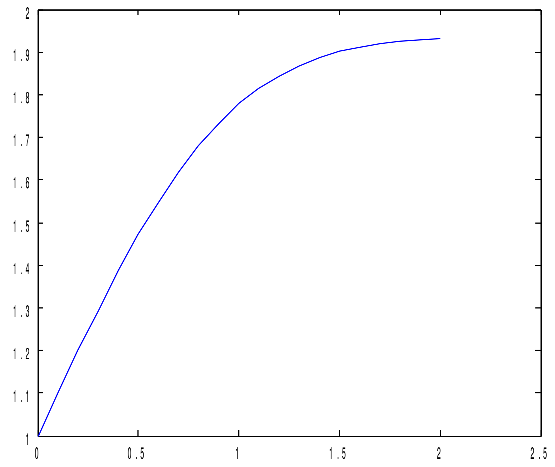
To execute the M-file **euler**, you need to define the function f= e^{-x^2} first as a vector function.

```
>> f=inline(vectorize('exp(-x^2)'),'x','y');
```

The given initial problem has x_0=0, y_0=1, x_n=2 and n=20. So, execute the M-file by

```
>> [x,y]=euler(f, 0, 1, 2, 20)
```

Note that in some cases the step size h, not the number of steps n may be given. Since h = $(x_n - x_0)$/n, you can calculate the number of steps n as n=$(x_n - x_0)$/h in those cases.



The outcome of the command will be a list of (x,y)-values. Similarly as for **ode45**, if you want your x and y values to be displayed in two columns next to each other (so that it is easy to see the y-value for a corresponding x-value), simply type

>> [x,y]
To graph this list, use **plot(x,y).**

The list consists of 21 (x,y) points. The list starts with the initial condition (x,y)=(0,1). The x-values are all h=(2-0)/20=0.1 units apart. The last x-value is 2 and the corresponding y-value is 1.9311. From this list you can see the y-values corresponding to x=1 and x=2.

x	y
1	1.7778
2	1.9311

Alternatively, if you need to display a specific (x,y)-value, you need to determine the i-value that corresponds to this point (i.e. you need to count the steps performed until the point has been calculated). Then you can display the point by typing x(i) and y(i).

For example, the point with x=1 is the 11th point calculated (Note: not 10th point - recall that x=0 corresponds to first point calculated, not the zeroth point calculated). So, to display this x-value you can type
>> x(11) and obtain the answer **ans = 1.000**
To display the corresponding y-value, you can type
>> y(11) and obtain the answer **ans = 1.7778**
Similarly, the point with x=2 is the 21th point calculated. You can display the x and y values as follows.
>> x(21) ans = 2.000
>> y(21) ans = 1.9311

<u>Practice problems</u>

1. a) Find the general solution of the differential equation $y'-2y=6x$.
   b) Find the particular solution with initial condition y(0)=3.
   c) Plot the particular solution on interval [0,2] and find the value of this solution at 2.

2. Graph the solutions of the differential equation $y'=0.1y(1-y)$ for the y-values of the initial condition $y(0)$ taking values 0.1, 0.3, 0.5 and 0.7.

3. Consider the autonomous equation $y' = (y+1)(3-y)^2(5-y)$.
   a) Find the equilibrium solution(s) of the equation and check the stability. Sketch the graph of all solutions.
   b) Use Matlab to sketch the direction field of this equation. Check if the graph agrees with your answer for part a).

4. Consider the equation $y' = (y+1)(3-y)^2(5-y)$. Using **ode45**, find the value of the solution with the initial condition y(0)=4 at x=5. Graph the solution with initial condition y(0)=4 for x-values in [0, 5].

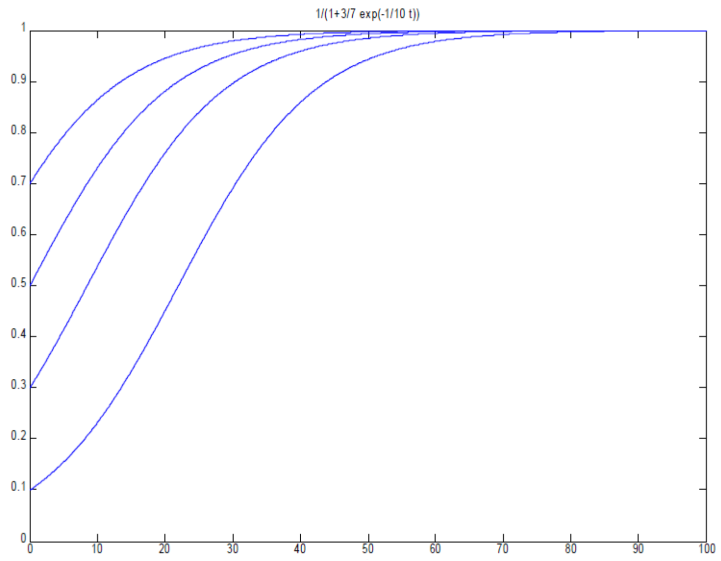
5. Consider the equation y' = (y-1)(5-y). Using the M-file **euler**, find the value of the solution with the initial condition y(0)=2 at x=3 for step size of 0.25. Graph the solution

with initial condition y(0)=2 for x-values in [0, 3].

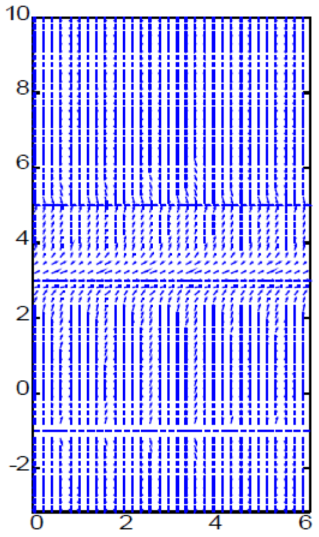
Solutions.

1. a) General solution: **dsolve('Dy-2*y=6*x', 'x')** **ans= (-6x-3)/2+C1*exp(2x)**
b) Particular solution: **dsolve('Dy-2*y=6*x', 'y(0)=3', 'x')** **ans= (-6x-3+9*exp(2x))/2**
c) **syms x** **ezplot((-6x-3+9*exp(2x))/2, [0 2])**
To find the value at 2: **f=inline((-6x-3+9*exp(2x))/2, x)** **f(2)**
ans=238.19

2. Modify the M-file **many_solutions** so that the second line is **cval=0.1:0.2:0.7** (this means that cval is taking values starting at 0.1 and ending at 0.7 that are 0.2 away from each other). Thus cval= 0.1, 0.3, 0.5 and 0.7. Also, to see better the limiting behavior of the solutions, you may want to graph the solutions on domain that includes large values of x. To do this modify the line with ezplot command to **ezplot(subs(s, 'c', cval), [0 100])** for example. Then execute **s = dsolve('Dy=0.1*y*(1-y)','y(0)=c', 'x')** and **many_solutions(s)** and obtain the following graph.

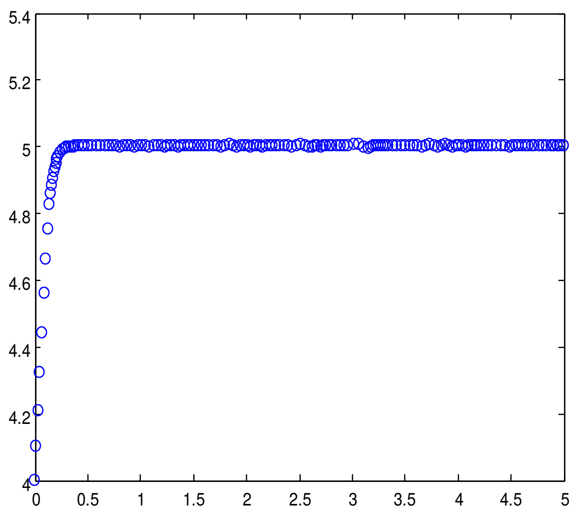


3. a) Set the right side of the equation equal to zero. Obtain the equilibrium solutions y=-1, y=3 and y=5. Analyze the sign. Obtain that y=-1 is unstable, y=3 is semistable and y=5 is stable.
b) First inline the function **f=inline(vectorize('(y+1)*(3-y)^2*(5-y)'), 'x', 'y')**



Then you can execute the relevant M-file by **direction_field(f)** To get a graph with all three equilibrium solutions displayed, you need to modify the meshgrid command so that y-values smaller than -1 and larger than 5 are displayed as well. For example, **[x,y]=meshgrid(0:0.2:6, -3:0.2:10);**

4. First, inline the function as a vector function using the same command as in previous problem. To display x and y values use **[x,y]=ode45(f, [0,5], 4)**. The last y-value corresponds to value at x=5 and it is **y=4.9997.** To get the graph of solution with the initial condition y(0)=4 on interval [0,5] use **ode45(f, [0,5], 4)**.



5. First, inline the function as a vector function using f=inline(vectorize('(y-1)*(5-y)'), 'x', 'y'). Note that the step size of 0.25 corresponds to the number of steps n=(3-0)/0.25=12.

Execute the M-file **euler** by **[x,y]=euler(f, 0, 2, 3, 12)**
To display the answers in two columns, one corresponding to x and the other to y-values use simply **[x,y]**. The list starts by x=0 and y=2. The list ends with x=3 and y=5.000. Graph the list by using **plot(x,y)**. The graph should look similar to the graph from the previous problem.

6. Second and Higher Order Differential Equations

Second order linear equations can be solved similarly as the first order differential equations: analytically using **dsolve** and numerically using **ode45**. For command **dsolve**, recall that we represent the first derivative of the function y with **Dy**. The second derivative of y is represented with **D2y**.

Example 6.1. a) Find general solution of $y''-3y'+2y = \sin x$.
b) Find the particular solution of the same equation with the initial conditions $y(0) = 1, y'(0)=-1$.

a) For the general solution, use: **>> dsolve('D2y-3*Dy+2*y=sin(x)', 'x')**
ans=3/10*cos(x)+1/10*sin(x)+C1*exp(x)+C2*exp(2*x)

b) For the particular solution with the initial conditions y(0) = 1, y'(0)=-1, use:
>> dsolve('D2y-3*Dy+2*y=sin(x)', 'y(0)=1', 'Dy(0)=-1', 'x')
ans = 3/10*cos(x)+1/10*sin(x)+5/2*exp(x)-9/5*exp(2*x).

For equations that can not be solved in terms of elementary functions, we use numerical methods. For ode45, the second order differential equation must be ***converted to a system of two first order equations*** using the substitution

$$\text{(S)} \qquad y=y_1 \text{ and } y'=y_2.$$

The first equation of the new system reflects the relation between the two new functions: the second one is the derivative of the first. So, the first equation is

$$\text{(1)} \qquad y_1'=y_2.$$

The second equation is obtained by applying the substitution to the original differential equation and solving it for $y''=y_2'$. For example, the equation $y''-3y'+2y = \sin x$ from the previous example with substitution (S) becomes $y_2'-3y_2 +2 y_1 = \sin x$. Solving for y_2' produces the second equation of the system to be $y_2'=\sin x+3y_2-2y_1$. Thus, the system is

$$\text{(1)} \quad y_1'=y_2 \qquad \text{and} \qquad \text{(2)} \quad y_2'=\sin x+3y_2-2y_1$$

We illustrate the use of **ode45** command for solving second order equations on the following example.

Example 6.2. Consider the following initial value problem

$$y'' + x\,y' + y = 0 \text{ with } y(0)=1 \text{ and } y'(0)=0.$$

(a) Using ode45, graph the solution on interval [0, 5].
(b) Using ode45, display the list of y-values of the solution for the integer x-values from 0 to 5.

First, you need to convert the given second order differential equation into a system of two first order equations using the substitution (S) $y=y_1$ and $y'=y_2$. The first equation of the new system is (1) $y_1'=y_2$. The second equation is obtained by using the substitution for the given equation $y'' + xy' + y = 0$ to obtain $y_2' + xy_2+ y_1 =0$ and then solving for y_2' to get (2) $y_2'=-xy_2- y_1$. Thus, the newly obtained system is

(1) $y_1'=y_2$ and (2) $y_2'=-x\, y_2- y_1$.

In Matlab, the two new functions can be denoted by **y(1)** and **y(2)**. The solution **y** will be represented as vector **y=[y(1); y(2)]**. Keep in mind that **y(1)** corresponds to the solution *y* of the original equation. The function **y(2)** corresponds to the derivative *y*'. This problem is not asking for the derivative *y*' so you can consider **y(2)** to be a byproduct.

Inline the right side of the two equations as a vector function **f** that depends on independent variable **x** and dependent variable **y**.

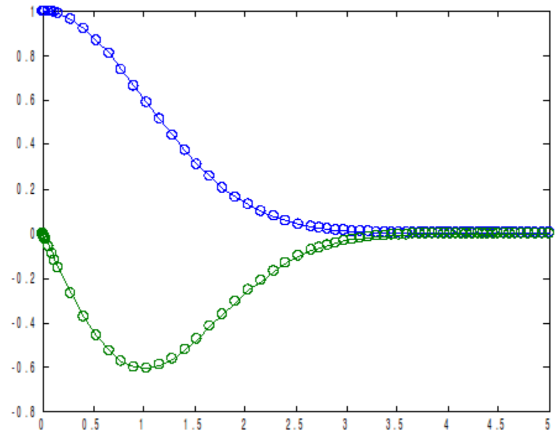
```
f=inline('[y(2); -x*y(2)-y(1)]','x','y');
```

The first entry of **f** is the right side of the first equation and the second entry of **f** is the right side of the second equation.

(a) To graph the solution on interval [0, 5], you can use

```
ode45(f, [0, 5], [1;0])
```

In this command, **[0, 5]** indicates the interval for *x* and **[1;0]** indicate the initial values *y*(0)=1 and *y*'(0)=0. The output is a graph with two functions **y(1)** representing the solution *y* and **y(2)** representing its derivative *y*'. The first one will be plotted in blue and the second one in green.



Alternatively, you can use

[x,y]=ode45(f, [0, 5], [1;0]); followed by **plot(x,y)**

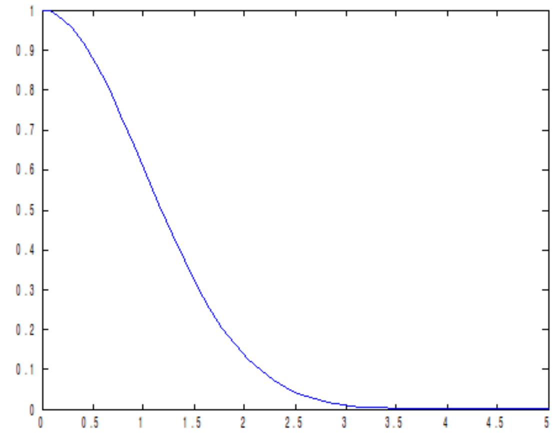
The first command calculates numerical values of the solution. Since we do not need those values displayed, the command ends with a semi-colon. The second commands plots the values calculated. The graph looks similar as the first graph above.

To display graph of y without the graph of *y*', you can use

[x,y]=ode45(f, [0, 5], [1;0]); followed by

plot(x,y(:,1))

These command produce the following graph.



(b) To obtain numerical values of the solution use

```
[x, y] = ode45(f, [0:1:5], [1;0])
```

Here **[0:1:5]** indicates that the x is taking values starting at 0, ending at 5 at step 1 away from each other. **[1;0]** indicates the initial values, just as in part (a). The vector **y** in the output will consist of two columns. The first **y(1)** consists of the values of the solution *y* at *x*-values between 0 and 5 and the second **y(2)** consists of the values of the derivative *y*' at these points. If you want the *x* and *y* values to be displayed in columns next to each other (so that it is easy to see the y-value for a corresponding *x*-value), you can use **[x,y].** Obtain the list

```
ans =
        0    1.0000    0
   1.0000    0.6065   -0.6065
   2.0000    0.1354   -0.2707
   3.0000    0.0111   -0.0333
   4.0000    0.0003   -0.0013
   5.0000    0.0000   -0.0000
```

Here the three columns represent *x, y* and *y*' values respectively.

Practice problems 2

1. a) Find the general solution of the equation $y''-4y'+4y=e^x+x^2$. b) Find particular solution of the initial value problem with $y(0)=8$, $y'(0)=3$.
2. Using **ode45** graph the solution of $y''+x^2y'+y=\cos 2x$, $y(0)=1$, $y'(0)=-1$ for t in [0, 4].

Solutions

1. a) **dsolve('D2y-4*Dy+4*y=exp(x)+x^2', 'x')**
ans = exp(x)+1/4*x^2+1/2*x+3/8+C1*exp(2*x)+C2*exp(2*x)*x
b) **dsolve('D2y-4*Dy+4*y=exp(x)+x^2','y(0)=8', 'Dy(0)=3', 'x')**
ans = exp(x)+1/4*x^2+1/2*x+3/8+53/8*exp(2*x)-47/4*exp(2*x)*x

2. Convert to a system using $y=y_1$ and $y'=y_2$. The first equation of the system is $y_1'=y_2$ and the second is obtained from $y''+x^2y'+y=\cos 2x \rightarrow y_2'+x^2y_2+y_1=\cos 2x \rightarrow y_2'=\cos 2x - x^2y_2 - y_1$. So the system is

$$(1)\ y_1'=y_2 \qquad \text{and} \qquad (2)\ y_2'=\cos 2x - x^2 y_2 - y_1$$

Inline the right side of the two equations as a vector function **f** by

f=inline('[y(2); cos(2*x)-x^2*y(2)-y(1)]','x','y').

To graph the solution on interval [0, 4], you can use **ode45(f, [0, 4], [1;-1])**. The values **[1;-1]** correspond to the y and y' values from the initial conditions.

7. Systems of Differential Equations

Symbolic Solutions. You can find the symbolic solutions of a system of differential equations by using the command **dsolve**.

Example 7.1. Consider the system

$$dx/dt=2x-y \qquad dy/dt=3x-2y$$

a) Find the general solution of this system.
b) Find the particular solution of the initial value problem with x(0)=1 and y(0)=2.
c) Graph the particular solution on interval 0≤t≤20.

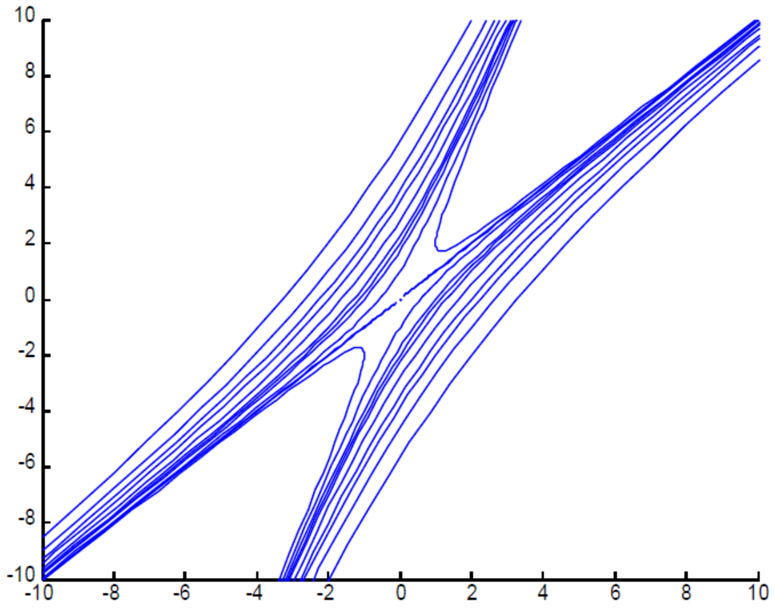
d) Plot sufficiently many solutions in the phase plane to determine the type of the equilibrium point (0,0).

a) **[x,y] = dsolve('Dx = 2*x - y', 'Dy = 3*x - 2*y', 't')**
b) **[x,y] = dsolve('Dx = 2*x - y', 'Dy = 3*x - 2*y', 'x(0) = 1', 'y(0) = 2', 't')**
c) To graph these two solutions on [0,20], you can use

ezplot(x, [0,20])   hold on   ezplot(y, [0,20])   hold off

d) To get the trajectories in the phase plane, you can graph a few solutions with different initial conditions on the same plot. For example, to graph the solutions for initial conditions x(0) and y(0) taking integer values between -2 and 2 and the value of parameter t between -3 and 3 taking 0.1 as a step size (thus, the t values will be: -3, -2.9, -2.8,...., 2.8, 2.9, 3) you can use the following M-file.

```
close all; axes; hold on
t = -3:0.1:3;
for a = -2:2
  for b = -2:2
    echo off
    [x,y] = dsolve('Dx = 2*x - y', 'Dy =
3*x - 2*y', 'x(0) = a', 'y(0) = b', 't');
       xv = inline(vectorize(x), 't', 'a',
'b');
       yv = inline(vectorize(y), 't', 'a',
'b');
    plot(xv(t, a, b), yv(t, a, b))
  end
end
hold off
axis([-10 10 -10 10])
```



From the graph, we conclude that (0,0) is unstable and that it is a saddle point.

Numeric Solutions. Finding symbolic solutions might be very limiting because many systems of differential equations cannot be solved explicitly in terms of elementary functions. For those equations or systems of equations, numerical methods are used in order to get the approximate solution. To find numeric solutions, you can use the command **ode45**. In order to use it, the system needs to be in the form x'=f(x,y,t) and y'=g(x,y,t) and the right sides of the equations should be represented as a vector function using **inline** first. The function x can be represented as **y(1)** and the function y as **y(2)**. The first entry of the inlined function **f** is the right side of the first equation and the second entry of **f** is the right side of the second equation.

<u>**Example 7.2**</u>. Consider the system

$$dx/dt=2x-x^2-xy \qquad dy/dt=xy-y$$

with the initial conditions $x(0)=1$ and $y(0)=2$.
a) Display the (x,y)-values of a numerical solution for t taking integer values between 0 and 6.

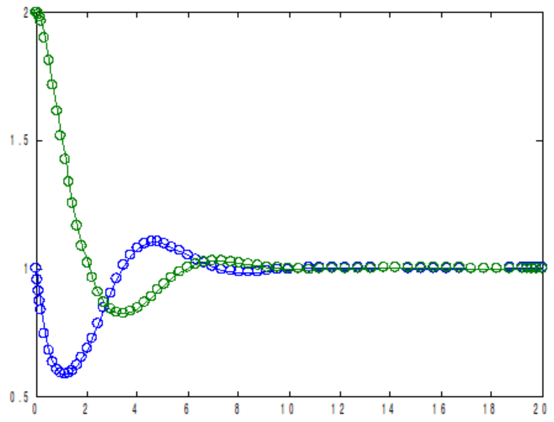
b) Graph the solutions on interval [0,20].
c) Graph the solution from b) in the phase plane.
d) Plot sufficiently many solutions of this system (without the given initial conditions) in the phase plane to determine the type of the equilibrium point (1,1).

First, inline the right side of equation as a function of independent variable *t*. The unknown functions *x* and *y* are represented by **y(1)** and **y(2)** respectively.

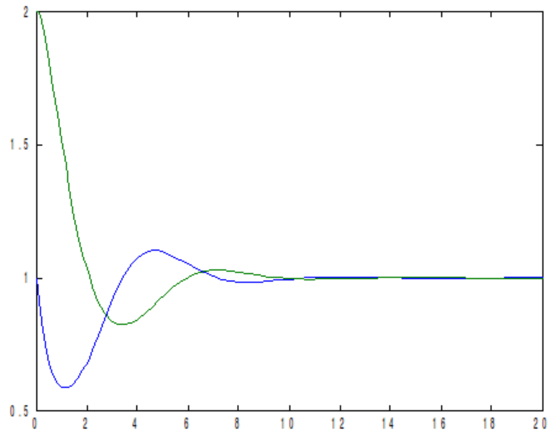
```
f = inline('[2*y(1)-y(1)^2-y(1)*y(2); y(1)*y(2)-y(2)]','t','y');
```

a) The command **[t,y]=ode45(f, [0:1:6], [1;2])**
computes the (*x,y*)-values at *t*=0,1,2,...,6. The part **[0:1:6]** indicates that *t*-values start at 0, end at 6 and are 1 step away from each other. The part **[1;2]** reflects the initial conditions *x*(0)=1 and *y*(0)=2. Note that here **y** is a vector whose entries will be the values of **y(1)** representing x and **y(2)** representing y. If you want the *t, x* and *y* values to be displayed in columns next to each other (so that it is easy to see the (*x,y*)-value for a corresponding *t*-value), you can use **[t,y].** Obtain the list

```
ans =
       0    1.0000   2.0000
  1.0000    0.5909   1.5148
  2.0000    0.6791   1.0294
  3.0000    0.9123   0.8370
  4.0000    1.0763   0.8424
  5.0000    1.0980   0.9281
  6.0000    1.0502   1.0016
```

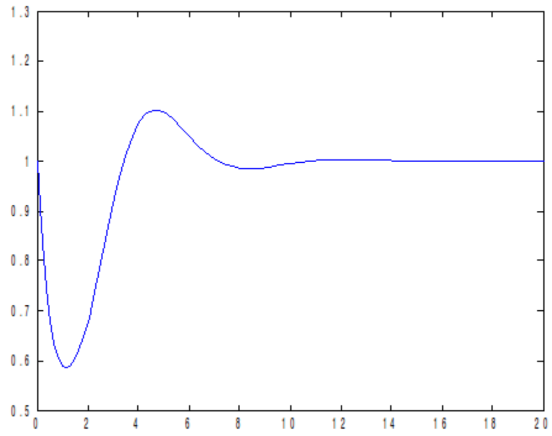


b) The command **ode45(f,[0,20],[1;2])** will graph the two solutions on the same plot as functions of *t*. The function *x* will be graphed in blue and *y* in green (the first graph).



Alternatively, you can graph using
[t,y]=ode45(f,[0 20],[1;2]); followed by **plot(t,y)**
The first command calculates numerical values of the solution. Since we do not need those values displayed, the command ends with a semi-colon. The second commands plots the values calculated. The graph (second one displayed) looks similar as the first graph above.

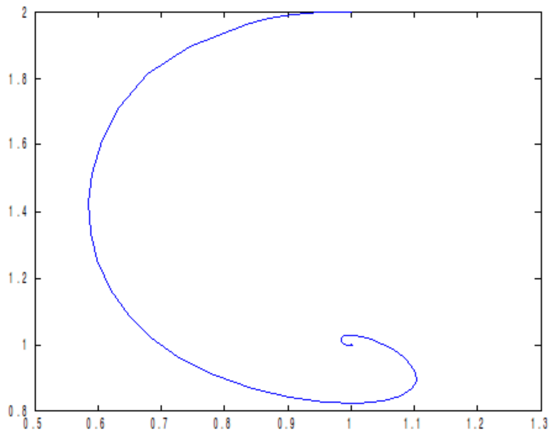
Note that both x and y approach 1 for large values of t. This may be relevant when determining the stability of the equilibrium point (1,1).



You may need to display just the first or just the second function. In this case, the command
[t,y]=ode45(f,[0 20],[1;2]); followed by **plot(t,y(:,1))**
plots just *x*-values (third graph displayed).

Similarly, the command
[t,y]=ode45(f,[0 20],[1;2]); followed by **plot(t,y(:,2))** plots just *y*-values.

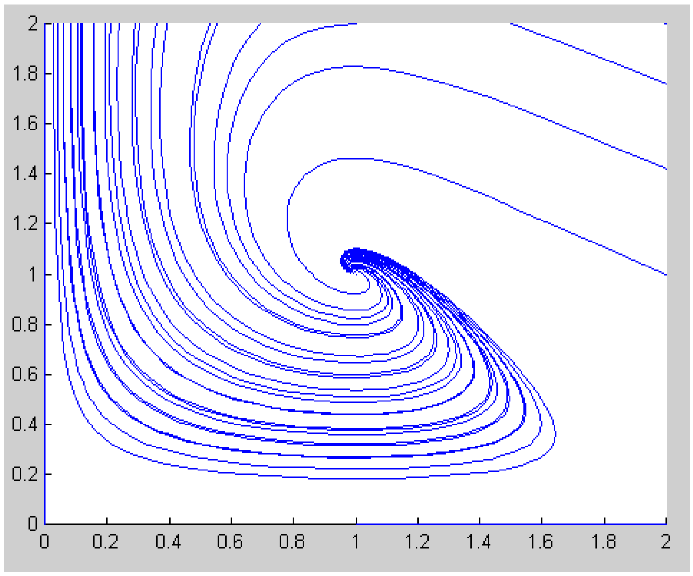
c) The command
[t,y] = ode45(f,[0,20],[1;2]); followed by
plot(y(:,1),y(:,2))
plots the solution in the phase plane: it plots (*x,y*) as a parametric curve of parameter *t* in (*x,y*)-plane.



The graph on the right displays the output of this command. Note that we can determine the **orientation** of this parametric curve using the previous graphs. In particular, from the previous graphs we can note that both x and y values converge to 1. Moreover, the initial condition (x,y)=(1,2) indicates that this is the initial point. Thus, the curve is traced starting from point (1,2) and ending at point (1,1).

d) The following M-file can be used to graph the trajectories in the phase plane for x(0) and y(0) taking integer initial values between 0 and 5.

close all; hold on
for a = 0:1:5
  for b = 0:1:5 (modify these values if necessary to change the density and position of the curves)
    [t, y] = ode45(f, 0:0.2:20, [a; b]);
    plot(y(:,1), y(:,2))
  end
end
hold off
axis([0 2 0 2]) (modify these values to change the window)



From the graph, we conclude that (1,1) is a spiral point. The stability can be determined by analyzing the graphs of solutions as functions of t: both x and y approach 1, thus (1,1) is stable.