Differential Equations
Lia Vas

Nonhomogeneous Linear Differential Equations. Methods. Applications

Consider a nonhomogeneous linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(x).$$

The general solution of such equation is of the form

$$y = y_h + y_p$$

where y_h is the general solution of homogeneous equation and y_p is called the **particular solution** and depends on the nonhomogeneous part. There are two main methods for finding a particular solutions of nonhomogeneous equations.

1. **Variation of parameters.** This method is completely general but sometimes tends to be messy.

2. **Undetermined coefficients**. This method is easier but works just when the function $g(x)$ is of a specific form and, thus, it is not general.

We shall present the methods for the case $n = 2$. Both methods can easily be generalized to higher orders.

Variation of parameters

Consider the equation $ay'' + by' + cy = g(x)$ and assume that y_1 and y_2 are solutions of the homogeneous part so that $y_h = c_1 y_1 + c_2 y_2$ is the general solution of the homogeneous part. The particular solution y_p is obtained by assuming that c_1 and c_2 are not constants but functions that depend on x. Since there is just one equation and we are introducing two new functions, we can impose one condition on them with no risk of loosing generality. Let us denote the two new functions by v_1 and v_2 so that

$$y_p = v_1 y_1 + v_2 y_2.$$

To find the unknown functions v_1 and v_2, find the derivatives of y_p.

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

and impose the condition that $v_1' y_1 + v_2' y_2 = 0$. Thus $y_p' = v_1 y_1' + v_2 y_2'$ and so

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Substituting derivatives in the equation and keeping in mind that y_1 and y_2 are solutions of homogeneous part, we obtain

$$av_1'y_1' + av_1y_1'' + av_2'y_2' + av_2y_2'' + bv_1y_1' + bv_2y_2' + cv_1y_1 + cv_2y_2 =$$

$$= v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) + av_1'y_1' + av_2'y_2' =$$

$$= av_1'y_1' + av_2'y_2' = g.$$

Thus, to determine the functions v_1 and v_2, we need to solve two equations

$$v_1'y_1 + v_2'y_2 = 0 \quad \text{and} \quad av_1'y_1' + av_2'y_2' = g$$

First, solve the equations algebraically for v_1' and v_2' and then obtain v_1 and v_2 by integrating.

Example. Solve the equation $y'' - y' - 2y = e^{3x}$.

Solution. The characteristic equation is $r^2 - r - 2 = 0$. The roots are 2 and -1, so that $y_1 = e^{2x}$, $y_2 = e^{-x}$ and the homogeneous solution is $y_h = c_1y^{2x} + c_2y^{-x}$. The particular solution will be in the form $y_p = v_1y^{2x} + v_2y^{-x}$. The two equations for the unknown functions are

$$v_1'e^{2x} + v_2'e^{-x} = 0 \quad \text{and} \quad v_1'2e^{2x} - v_2'e^{-x} = e^{3x}$$

Eliminating v_2' gives us that $3v_1'e^{2x} = e^{3x}$ and so $v_1' = \frac{1}{3}e^x$. Integrating we obtain that $v_1 = \frac{1}{3}e^x$.

Solving the first equation for v_2' gives us $\frac{1}{3}e^{3x} = -v_2'e^{-x}$ and so $v_2' = -\frac{1}{3}e^{4x}$. Integrating we get $v_2 = -\frac{1}{12}e^{4x}$. This gives a particular solution $y_p = \frac{1}{3}e^xe^{2x} - \frac{1}{12}e^{4x}e^{-x} = \frac{1}{3}e^{3x} - \frac{1}{12}e^{3x} = \frac{1}{4}e^{3x}$. The general solution of the differential equation is

$$y = c_1e^{2x} + c_2e^{-x} + \frac{1}{4}e^{3x}.$$

Practice Problems. Solve the differential equations.

1. $y'' - 6y' + 9y = x^{-3}e^{3x}$.
2. $y'' - 5y' + 6y = 2e^x$
3. $y'' + 4y' + 4y = x^{-2}e^{-2x}$

Solutions.

1. $y_h = c_1e^{3x} + c_2xe^{3x}$ and $y_p = \frac{1}{2}x^{-1}e^{3x}$. The general solution is $y = c_1e^{3x} + c_2xe^{3x} + \frac{1}{2}x^{-1}e^{3x}$.
2. $y_h = c_1e^{2x} + c_2e^{3x}$ and $y_p = e^x$. The general solution is $y = c_1e^{2x} + c_2e^{3x} + e^x$.
3. $y_h = c_1e^{-2x} + c_2xe^{-2x}$ and $y_p = -\ln x\ e^{-2x} - e^{-2x}$. The general solution is $y = (c_1 - 1)e^{-2x} + c_2xe^{-2x} - \ln x\ e^{-2x} = C_1e^{-2x} + c_2xe^{-2x} - \ln x\ e^{-2x}$.

Undetermined Coefficients

Consider again a nonhomogeneous linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(x).$$

Recall that the general solution of such equation is of the form

$$y = y_h + y_p$$

where y_h is the general solution of homogeneous equation and y_p is the particular solution. The method of undetermined coefficients determines the particular solution y_p in the following cases

Case 1 $g(x)$ is a product of a polynomial and exponential function.

Case 2 $g(x)$ is a product of a polynomial, exponential function and a trigonometric function.

In particular, let $p_k(x)$ be a polynomial $a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0$ and p and q real numbers.

Case 1 If $g(x) = p_k(x) e^{px}$, then

$$y_p = x^s (A_k x^k + A_{k-1} x^{k-1} + \ldots + A_0) e^{px}$$

where s is the number of times p appears on the list of zeros of the characteristic equation and $A_0, \ldots, A_k$ are undetermined coefficients.

Case 2 $g(x) = p_k(x) e^{px} \cos qx$ or $g(x) = p_k(x) e^{px} \sin qx$, then

$$y_p = x^s (A_k x^k + A_{k-1} x^{k-1} + \ldots + A_0) e^{px} \cos qx + x^s (B_k x^k + B_{k-1} x^{k-1} + \ldots + B_0) e^{px} \sin qx$$

where s is the number of times $p + iq$ appears on the list of zeros of the characteristic equation and $A_0, \ldots, A_k$ and $B_0, \ldots, B_k$ are undetermined coefficients.

To find the undetermined coefficients, plug the particular solution and its derivatives into the original equation and determine the coefficients from there by equating the polynomials (on the same way as when solving partial fractions in a Calculus 2 course).

If $g(x)$ is a sum of functions $g(x) = g_1(x) + g_2(x) + \ldots + g_m(x)$ and each function $g_1(x)$, $g_2(x), \ldots,$ $g_m(x)$ is a function described under two cases above, then the particular solution y_p is the sum of particular solutions

$$y_p = y_{p1} + y_{p2} + \ldots + y_{pm}$$

where each solution y_{pi}, $i = 1, \ldots, m$ is obtained as in Case 1 or 2 described above.

Example 1. Solve the differential equation $y'' - 3y' - 4y = 12e^{2x}$.

Solution. The characteristic equation is $0 = r^2 - 3r - 4 = (r-4)(r+1)$ so 4 and -1 are zeros and $y_h = c_1 e^{4x} + c_2 e^{-x}$. The function $g(x)$ is $12e^{2x}$ and so the polynomial $p_k(x)$ is constant. Since 2 is not a zero of the characteristic equation $s = 0$ and so a particular solution is of the form

$$y_p = Ae^{2x}$$

Finding the derivatives $y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$ and substituting them into the equation yields

$$4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 12e^{2x}$$

Thus $4A - 6A - 4A = 12$ and so $-6A = 12$ giving us that $A = -2$. So, $y_p = -2e^{2x}$ and the general solution is $y = c_1e^{4x} + c_2e^{-x} - 2e^{2x}$.

Example 2. Solve the differential equation $y'' - 3y' - 4y = 8\sin x$.

Solution. The homogeneous part is the same as in the previous example and so $y_h = c_1e^{4x}+c_2e^{-x}$. The function $g(x)$ is $8\sin x$ and so the polynomial $p_k(x) = 8$ is constant. The exponential function is trivial $1 = e^{0x}$ and so $a = 0$ and the trigonometric function $2\sin x$ determines that $b = 1$. Then we are checking if $a + ib = 0 + 1i = i$ is a zero of the characteristic equation. Since it is not, $s = 0$. Thus we look for a particular solution in the form

$$y_p = A\cos x + B\sin x$$

Finding the derivatives $y_p' = -A\sin x + B\cos x$ and $y_p'' = -A\cos x - B\sin x$ and substituting them into the equation yields

$$-A\cos x - B\sin x + 3A\sin x - 3B\cos x - 4A\cos x - 4B\sin x = 8\sin x$$

Equate the terms with $\cos x$ and the terms with $\sin x$. This yields two equations in two unknowns.

$$-A - 3B - 4A = 0 \quad \text{and} \quad -B + 3A - 4B = 8$$

From the first equation $B = -5A/3$ and from the second $3A + 25A/3 = 8$ so $A = 12/17$. Thus $B = -20/17$ and so $y_p = \frac{12}{17}\cos x - \frac{20}{17}\sin x$ and the general solution is $y = c_1e^{4x}+c_2e^{-x}+\frac{12}{17}\cos x-\frac{20}{17}\sin x$.

Example 3. Solve the differential equation $y'' - 3y' - 4y = 12e^{2x} + 8\sin x$.

Solution. Consider the function $g(x)$ as the sum of two separate parts: $12e^{2x}$ and $8\sin x$. Then look for the particular solution in the form $y_{p1} + y_{p2}$ where the particular solution y_{p1} is determined by the function $12e^{2x}$ and the particular solution y_{p2} is determined by the function $8\sin x$. From previous two examples, $y_{p1} = -2e^{2x}$ and $y_{p2} = \frac{12}{17}\cos x - \frac{20}{17}\sin x$. Thus, the general solution is $y = c_1e^{4x} + c_2e^{-x} - 2e^{2x} + \frac{12}{17}\cos x - \frac{20}{17}\sin x$.

Example 4. Solve the differential equation $y'' - 3y' - 4y = 5xe^{4x}$.

Solution. From previous examples, 4 and -1 are zeros of characteristic equation. The function $g(x)$ is $5xe^{4x}$ and so the polynomial $p_k(x)$ is a linear polynomial. Since 4 is a simple zero of the characteristic equation $s = 1$ and so a particular solution is of the form

$$y_p = x^1(Ax + B)e^{4x} = (Ax^2 + Bx)e^{4x}$$

Finding the derivatives $y_p' = (2Ax + B)e^{4x} + 4(Ax^2 + Bx)e^{4x} = (4Ax^2 + 4Bx + 2Ax + B)e^{4x}$ and $y_p'' = (8Ax + 4B + 2A)e^{4x} + 4(4Ax^2 + 4Bx + 2Ax + B)e^{4x} = (16Ax^2 + 16Ax + 16Bx + 8B + 2A)e^{4x}$ and substituting them into the equation yields

$$(16Ax^2 + 16Ax + 16Bx + 8B + 2A - 12Ax^2 - 12Bx - 6Ax - 3B - 4Ax^2 - 4Bx)e^{4x} = 5xe^{4x}$$

Thus $16Ax^2 + 16Ax + 16Bx + 8B + 2A - 12Ax^2 - 12Bx - 6Ax - 3B - 4Ax^2 - 4Bx = 5x$. Equating the similar terms of the polynomials on the left and right side yields two equations in two unknowns

$$+16A + 16B - 12B - 6A - 4B = 5 \text{ and } 8B + 2A - 3B = 0$$

Thus $10A = 5$ and $5B = 2A$ giving us that $A = 1/2$ and $B = 1/5$. So, $y_p = x(\frac{1}{2}x + \frac{1}{5})e^{4x}$ and the general solution is $y = c_1e^{4x} + c_2e^{-x} + x(\frac{1}{2}x + \frac{1}{5})e^{4x}$.

Practice Problems. Find the general solutions of problems 1 – 6.

In problems 7 – 9, find the *form* of particular solutions and the general solutions. You **do not** have to solve for unknown coefficients in particular solutions.

1. $y'' + y = 3e^{-x}$
2. $y'' - 5y' - 6y = 4e^{2x}$
3. $y'' - 5y' + 6y = 4e^{2x}$
4. $y'' + 4y = 5x^2e^x$
5. $y'' - 2y' + y = 7xe^x$
6. $y'' + 2y' - 3y = 5\sin 3x$
7. $y'' - 3y' - 10y = 3xe^{2x} + 5e^{-2x}$
8. $y'' - 8y' + 16y = 3x^2 - 5e^{4x}$
9. $y'' + 4y' + 13y = -2\sin 3x + e^{-2x}\cos 3x$

Solutions.

1. $r = \pm i$ are zeros of the characteristic equation and $y_h = c_1\cos x + c_2\sin x$. Then $s = 0$ since -1 is not a zero of the characteristic equation, and so $y_p = Ae^{-x}$. Determine A to be $\frac{3}{2}$ so $y_p = \frac{3}{2}e^{-x}$ and the general solution is $y = c_1\cos x + c_2\sin x + \frac{3}{2}e^{-x}$.

2. $r = 6$ and $r = -1$ are zeros of the characteristic equation and $y_h = c_1e^{6x} + c_2e^{-x}$. Then $s = 0$ since 2 is not a zero of the characteristic equation, and so $y_p = Ae^{2x}$. Determine A to be $-\frac{1}{3}$ so $y_p = -\frac{1}{3}e^{2x}$ and the general solution is $y = c_1e^{6x} + c_2e^{-x} - \frac{1}{3}e^{2x}$

3. $r = 2$ and $r = 3$ are zeros of the characteristic equation and $y_h = c_1e^{2x} + c_2e^{3x}$. Then $s = 1$ since 2 is a (single) zero of the characteristic equation, and so $y_p = Axe^{2x}$. Determine A to be -4 so $y_p = -4xe^{2x}$ and the general solution is $y = c_1e^{2x} + c_2e^{3x} - 4xe^{2x}$.

4. $r = \pm 2i$ are zeros of the characteristic equation and $y_h = c_1\cos 2x + c_2\sin 2x$. Then $s = 0$ since 1 is not a zero of the characteristic equation, and so $y_p = x^0(Ax^2 + Bx + C)e^x$. Obtain that $A = 1, B = -\frac{4}{5}$, and $C = -\frac{2}{25}$ so $y_p = (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$ and the general solution is $y = c_1\cos 2x + c_2\sin 2x + (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$.

5. $r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0$ so $r = 1$ is a double zero. So $y_h = c_1e^x + c_2xe^x$. Then $s = 2$ since 1 is double zero of the characteristic equation, and so $y_p = x^2(Ax+B)e^x = (Ax^3+Bx^2)e^x$. Obtain that $A = \frac{7}{6}$, and $B = 0$ so $y_p = \frac{7}{6}x^3e^x$ and the general solution is $y = c_1e^x + c_2xe^x + \frac{7}{6}x^3e^x$.

6. $r = -3$ and $r = 1$ are zeros of the characteristic equation and $y_h = c_1e^{-3x} + c_2e^x$. Then $s = 0$ since $3i$ is not a zero of the characteristic equation, and so $y_p = A\cos 3x + B\sin 3x$. Obtain that $A = -\frac{1}{6}$ and $B = -\frac{1}{3}$. So $y_p = -\frac{1}{6}\cos 3x - \frac{1}{3}\sin 3x$. The general solution is $y = c_1e^{-3x} + c_2e^x - \frac{1}{6}\cos 3x - \frac{1}{3}\sin 3x$.

7. The roots of the characteristic equation $r^2 - 3r - 10 = (r-5)(r+2) = 0$ are 5 and -2 so the homogeneous solution is $y_h = c_1 e^{5x} + c_2 e^{-2x}$.

   You have to consider functions $g_1(x) = 3xe^{2x}$ and $g_2(x) = 5e^{-2x}$ separately and obtain two separate particular solutions y_{p1} and y_{p2}.

   For $g_1(x) = 3xe^{2x}$, note that 2 is not a solution of the characteristic equation so $s = 0$. Because of the linear polynomial $3x$ in $g_1(x)$, the first particular solution has the form $y_{p1} = (Ax+B)e^{2x}$.

   For $g_2(x) = 5e^{-2x}$, note that -2 is a solution of the characteristic equation so $s = 1$. Because of the constant polynomial 5 in $g_2(x)$, the second particular solution has the form $y_{p1} = x^1 Ce^{-2x} = Cxe^{-2x}$.

   The general solution has the form $y = c_1 e^{5x} + c_2 e^{-2x} + (Ax+B)e^{2x} + Cxe^{-2x}$.

8. The characteristic equation $r^2 - 8r + 16 = (r-4)(r-4) = 0$ has a double zero $r = 4$. The homogeneous solution is $y_h = c_1 e^{4x} + c_2 xe^{4x}$. Similarly to the previous problem, you have to consider functions $g_1(x) = 3x^2$ and $g_2(x) = -5e^{4x}$ separately and obtain two separate particular solutions y_{p1} and y_{p2}.

   For $g_1(x) = 3x^2 = 3x^2 e^{0x}$, note that 0 is not a solution of the characteristic equation so $s = 0$. Because of the quadratic polynomial $3x^2$ in $g_1(x)$, the first particular solution has the form $y_{p1} = Ax^2 + Bx + C$.

   For $g_2(x) = -5e^{4x}$, note that 4 is a double solution of the characteristic equation so $s = 2$. Because of the constant polynomial -5 in $g_2(x)$, the second particular solution has the form $y_{p1} = x^2 De^{4x} = Dx^2 e^{4x}$.

   The general solution has the form $y = c_1 e^{4x} + c_2 xe^{4x} + Ax^2 + Bx + C + Dx^2 e^{4x}$.

9. The characteristic equation $r^2 + 4r + 13 = 0$ has solutions $r = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$. So, the homogeneous solution is $y_h = c_1 e^{-2x} \cos 3x + c_2 e^{-2x} \sin 3x$.

   For $g_1(x) = -2 \sin 3x = -2e^{0i} \sin 3x$, note that $0 \pm 3i$ are not solutions of the characteristic equation so $s = 0$. Because of the constant polynomial -2 in $g_1(x)$, the first particular solution has the form $y_{p1} = A \cos 3x + B \sin 3x$.

   For $g_2(x) = e^{-2x} \cos 3x$, note that $-2 \pm 3i$ is a solution of the characteristic equation so $s = 1$. Because of the constant polynomial 1 in $g_2(x)$, the second particular solution has the form $y_{p1} = x(Ce^{-2x} \cos 3x + De^{-2x} \sin 3x) = Cxe^{-2x} \cos 3x + Dxe^{-2x} \sin 3x$.

   The general solution has the form $y = c_1 e^{-2x} \cos 3x + c_2 e^{-2x} \sin 3x + A \cos 3x + B \sin 3x + Cxe^{-2x} \cos 3x + Dxe^{-2x} \sin 3x$.

Applications

Many physical processes can be modeled by linear differential equations. For example, mechanical oscillations, electric circuits and more.

Mechanical oscillations. Consider a mass m on a spring. Let $u(t)$ denotes the position at time t. The following forces act on the mass:

1. The gravitational force mg,

2. The spring force F_s that is proportional to the natural length L plus any additional elongation $u(t)$, so $F_s = -k(L+u)$ By Hooke's law, $mg = kL$ where k is a spring constant

$$k = \frac{mg}{L}.$$

3. The damping or resistive force F_d that may arise because of resistance from the air, internal energy dissipation, friction between the mass and possible guides etc. It is proportional to the speed of the mass $F_d = \pm\gamma u'(t)$. The constant γ is called the damping constant, and the sign $\pm$ depends on the choice of the coordinate system for the motion (recall the problem with a falling object from the First Order Diff. Eq. handout – the same consideration applies here). We can choose the coordinate system so that this force acts in the opposite direction from mg and so $F_d = -\gamma u'(t)$.

4. A possible external force $F(t)$.

The total force $F = ma = mu''$, the product of the mass and the acceleration, is equal to the sum of all four acting forces

$$mu'' = mg - k(L+u) - \gamma u' + F(t)$$

since $mg - kL = 0$, we have the differential equation describing the motion:

$$mu'' + \gamma u' + ku = F(t)$$

If $\gamma = 0$, the oscillations are said to be **undamped**, otherwise they are **damped**.
If $F(t) = 0$, the oscillations are said to be **free**, otherwise they are **forced.**

Undamped free oscillations. The equation of motion for undamped free oscillations is

$$mu'' + ku = 0$$

Note that the characteristic equation of this differential equation is $mr^2 + k = 0$ and has solutions $r = \pm\sqrt{\frac{k}{m}}\, i$. If we denote $\sqrt{\frac{k}{m}}$ by ω_0, the general solution of this equation is

$$u = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

The constant ω_0 is called the **natural frequency** the constant $\frac{2\pi}{\omega_0}$ represents the **period** of the motion.

If we put $c_1 = R\cos\delta$ and $c_2 = R\sin\delta$, then R is the **amplitude**, δ is the **phase** and the solution is

$$u = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t = R\cos(\omega_0 t - \delta).$$

Note that for this type of oscillations, the amplitude is the same regardless of the time passed. The graph on Figure 1 represents a graph of this type of motion.

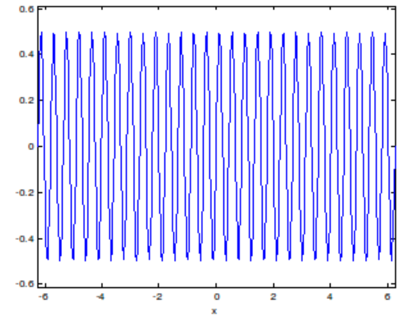

Figure 1: Undamped Free Oscillations

Damped free oscillations. The equation of motion for damped free oscillations is

$$mu'' + \gamma u' + ku = 0$$

The solutions of the characteristic equations are $r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$
There are three cases.

i) If $\gamma^2 - 4mk > 0$, the solutions are real, different and negative (because $\gamma^2 - 4mk < \gamma^2$). So, the solution is $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. Note that since $r_1, r_2 < 0$, the limit of u is zero when $t \to \infty$. In this case, the mass goes back to original position and does not oscillate. This motion is said to be **overdamped**.

ii) $\gamma^2 - 4mk = 0$, the solutions are real, equal ($r_1 = r_2$) and negative. The solution is $u = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$. Note that again, the limit of u is zero when $t \to \infty$. In this case, the mass also does not oscillate. The value of γ that makes $\gamma^2 - 4mk = 0$ is called the **critical damping.**

In these two cases, there are no oscillations. Figure 2 displays examples of overdamped and critically damped oscillations.

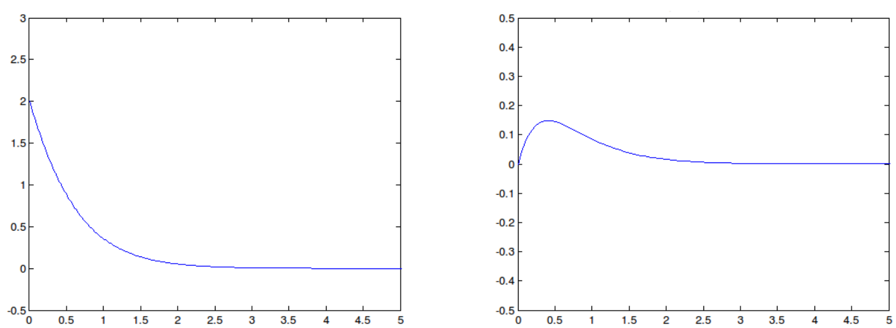

Figure 2: Overdamped Case

iii) $\gamma^2 - 4mk < 0$, the solutions are complex $r_1, r_2 = \frac{-\gamma}{2m} \pm i\frac{\sqrt{4km - \gamma^2}}{2m}$. So, the solutions is $u = e^{-\gamma t/(2m)}(c_1\cos\mu t + c_2\sin\mu t)$ where $\mu = \frac{\sqrt{4km - \gamma^2}}{2m}$. The presence of periodic functions in the solution indicates the oscillations.

This case occurs always when the damping is relatively small (i.e. $\gamma < \sqrt{4mk}$). This case is being referred to as **underdamping**. Figure 3 represents the graph of an underdamped oscillator.

If we put $c_1 = R\cos\delta$ and $c_2 = R\sin\delta$, the amplitude is given by the function

$$Re^{-\gamma t/(2m)}$$

Note that the amplitude converges to zero when $t \to \infty$. So, the mass oscillates about the original position but the oscillations are getting smaller and smaller as time passes by. The parameter μ is called the **quasi frequency** and $\frac{2\pi}{\mu}$ is called the **quasi period**.

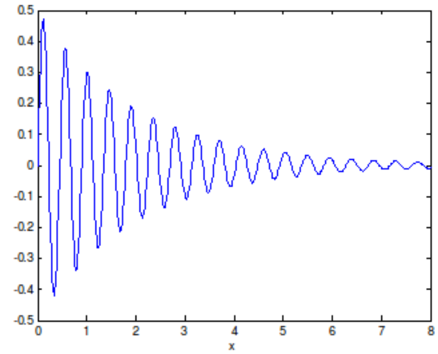

Figure 3: Underdamped Case

Undamped forced oscillations. The equation of motion for undamped forced oscillations is

$$mu'' + ku = F(t)$$

If the force F is periodic, we can write it as $F = F_0 \cos\omega t$ (or $F_0 \sin\omega t$). Recall that the characteristic equation has solutions $\pm\sqrt{\frac{k}{m}}i = \pm\omega_0 i$ so that the homogeneous solution is $u_h = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$.

The particular solution can be found using the Undetermined Coefficients method. The particular solution has the form

$$u_p = t^s(A\cos\omega t + B\sin\omega t)$$

where

- $s = 0$ if ωi is not a solution of the characteristic equation i.e $\omega \neq \omega_0$ and
- $s = 1$ if ωi is a solution of the characteristic equation i.e $\omega = \omega_0$.

If $\omega_0 \neq \omega$, the general solution has the form

$$u = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + A\cos\omega t + B\sin\omega t$$

A function of this form is a periodic function with periodic amplitude. The graph of one such function is on Figure 4. This type of motion is known as oscillations with **beats**.

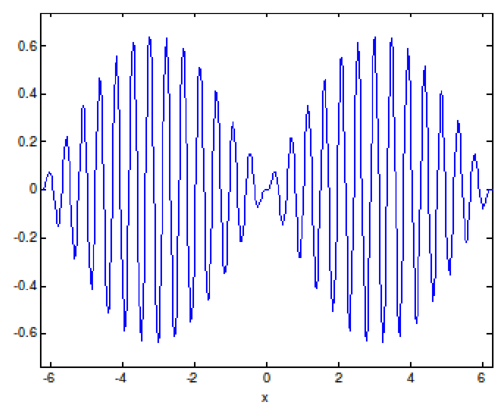

Figure 4: Beats

If $\omega_0 = \omega$, the frequency of the force is the same as the natural frequency. In this case, the general solution has the form

$$u = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + t(A\cos\omega t + B\sin\omega t)$$

Because of the term t which multiplies the trigonometric functions in the particular solution, the amplitude of the solution increases when $t \to \infty$. Thus, a function of this form is a periodic function with increasing amplitude. The graph of one such function is on Figure 5. This type of motion is known as oscillations with a **resonance**.

Mechanical resonance may cause swaying motions leading to a catastrophic failure of structures such as bridges, buildings, and vehicles. To prevent this from happening, such objects should be designed so that the mechanical resonance frequencies of the component parts do not match vibrational frequencies of any oscillating parts. Acoustic resonance is important when designing instruments. Like mechanical resonance, acoustic resonance can result in catastrophic failure of the object at resonance, such as breaking a glass with sound.

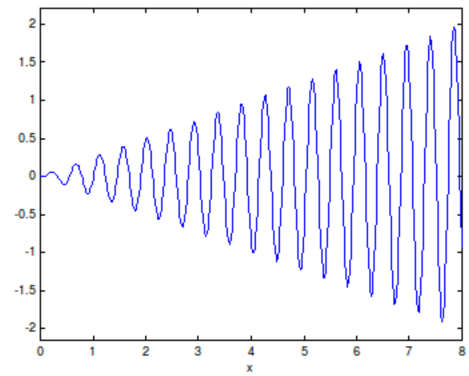

Figure 5: Resonance

Electric circuits. Consider an electric circuit with the resistance R, the capacitance C and the inductance L containing a battery producing the voltage $E(t)$ at time t. The current I and the charge Q are related by $I = \frac{dQ}{dt}$. The second Kirchhoff's law tells us that the applied voltage $E(t)$ is equal to the sum of voltage drops in the rest of the circuit. Since

- The voltage drop across the resistor is IR,
- The voltage drop across the capacitor is $\frac{Q}{C}$, and
- The voltage drop across the inductor is $L\frac{dI}{dt}$,

the following equation models this set up.

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

Since $I = \frac{dQ}{dt}$, $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$ and so we have a second order linear differential equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

The analysis of this equation is completely analogous to the analysis of the equation of mechanical motion $mu'' + \gamma u' + ku = F(t)$.

Hyperbolic Sine and Cosine. In many cases, the solutions of differential equations are represented in terms of hyperbolic sine and cosine rather than in terms of exponential functions. The hyperbolic sine and cosine are defined as

$$\sinh t = \frac{e^t - e^{-t}}{2} \qquad \text{and} \qquad \cosh t = \frac{e^t + e^{-t}}{2}$$

The name "hyperbolic" comes from the fact that $(\cosh t, \sinh t)$ form a hyperbola, analogously to the fact that the points $(\cos t, \sin t)$ form a circle.

Using the definitions of the hyperbolic functions, the following identities can be obtained.

$$\sinh t + \cosh t = e^t \quad \text{and} \quad \cosh t - \sinh t = e^{-t}$$

Thus,

$$\sinh at + \cosh at = e^{at} \quad \text{and} \quad \cosh at - \sinh at = e^{-at}$$

Using the hyperbolic functions, we can see the solutions of the equation $y'' - a^2 y = 0$ as completely analogous to $y'' + a^2 y = 0$, where a is positive. Let us compare these solutions.

Recall that the equation $y'' + a^2 y = 0$ has characteristic roots $\pm ai$ yielding the general solution $y = c_1 \cos at + c_2 \sin at$. The equation, $y'' - a^2 y = 0$ has characteristic roots $\pm a$ yielding the general solution $y = c_1 e^{at} + c_2 e^{-at}$. Represent this solution using hyperbolic functions and the above identities: $y = c_1(\sinh at + \cosh at) + c_2(\cosh at - \sinh at) = (c_1 + c_2)\cosh at + (c_1 - c_2)\sinh at$. Denoting $C_1 = c_1 + c_2$ and $C_2 = c_1 - c_2$, we obtain the solution in the form

$$y = C_1 \cosh at + C_2 \sinh at$$

that parallels the solutions $y = c_1 \cos at + c_2 \sin at$ of $y'' + a^2 y = 0$.

Converting Higher Order Equations into Systems of First Order Equations

Recall that a system of n first order differential equations has the form

$$y_1' = F_1(t, y_1, \ldots, y_n), \;\; y_2' = F_2(t, y_1, \ldots, y_n), \;\; \ldots, \;\; y_n' = F_n(t, y_1, \ldots, y_n),$$

Every **differential equation of order** n can be converted into a **system of** n **first order equations**. Thus, studying systems encompasses the study of higher order differential equations as well. In particular, finding numerical solution of higher order equations using Matlab command **ode45** requires this procedure.

A general n-th order differential equation $F(y^{(n)}, y^{(n-1)}, \ldots, y', y, t) = 0$ can be converted into a system of n differential equations of the first order in unknown functions $y_1, y_2, \ldots, y_n$ by considering

the substitution $\quad y_1 = y, \quad y_2 = y' = y_1', \quad y_3 = y'' = y_2', \ldots, \quad y_n = y^{(n-1)} = y_{n-1}'$.

The $n-1$ equations above starting from the second to the last one represent $n-1$ equations of the new first order system. The n-th equation of the system is obtained from the original equation which, using the substitution becomes

$$F(y_n', y_n, y_{n-1}, \ldots, y_2, y_1, t) = 0.$$

If solving for y_n' produces the equation $y_n' = f(x_n, x_{n-1}, \ldots, x_2, x_1, t)$, this becomes the n-th equation of the new system. So, the **new system of** n **first order equations** is the following.

$$\begin{array}{lcl} y_1' & = & y_2 \\ y_2' & = & y_3 \\ & & \ldots \\ y_{n-1}' & = & y_n \\ y_n' & = & = f(y_n, y_{n-1}, \ldots, y_2, y_1, t) \end{array}$$

Example. Convert the following differential equations into a system of first order equations.

1. $y'' - ty' + 7y = \sin t + t^2$
2. $y''' + 3y' - 2y = e^t$

Solution. (1) We need to convert the given second order differential equation into a system of two first order equations. The substitution $y_1 = y$ and $y_2 = y'$ converts the given equation in y into a system in y_1 and y_2. The two new variables are related by $y_1' = y_2$ and this relation is the first equation of the new system. With this substitution the given equation becomes $y_2' - ty_2 + 7y_1 = \sin t + t^2 \Rightarrow y_2' = ty_2 - 7y_1 + \sin t + t^2$ and this last equation is the second equation of the new system. So, the new system is

$$y_1' = y_2, \quad y_2' = ty_2 - 7y_1 + \sin t + t^2.$$

(2) We need to convert the given third order differential equation into a system of three first order equations. The substitution $y_1 = y$, $y_2 = y'$, and $y_3 = y''$ converts the given equation in y into a system in y_1, y_2, and y_3. The three new variables are related by $y_1' = y_2$ and $y_2' = y_3$ these two relations are the first two equations of the new system. With this substitution the given equation becomes $y_3' + 3y_2 - 2y_1 = e^t \Rightarrow y_3' = -3y_2 + 2y_1 + e^t$ and this last equation is the third equation of the new system. So, the new system is

$$y_1' = y_2, \quad y_2' = y_3, \quad \text{and} \quad y_3' = -3y_2 + 2y_1 + e^t.$$

Practice Problems.

1. Consider harmonic oscillations described by each of the following equations. In each case below, solve the equation, graph the solution and explain the type of motion that the graph displays.
   (a) $u'' + u = 0$, $u(0) = 1$, $u'(0) = 0$;
   (b) $u'' + u = \frac{1}{2}\cos 0.8t$, $u(0) = 0$, $u'(0) = 0$;
   (c) $u'' + u = \frac{1}{2}\cos t$, $u(0) = 0$, $u'(0) = 0$;
   (d) $u'' + \frac{1}{4}u' + u = 0$, $u(0) = 1$, $u'(0) = 0$;
   (e) $u'' + 2u' + u = 0$, $u(0) = 1$, $u'(0) = 0$;
2. Determine the values of γ for which the equation $u'' + \gamma u' + 9u = 0$ has solutions which are not overdamped.
3. A mass of 0.1 kg stretches a spring 0.05 m. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec, and if there is no damping, determine the position u as the function of time t. Graph the solution and explain the type of motion that the graph displays. Find the time when the mass first returns to its equilibrium position, the period and the frequency.
4. A series circuit has capacitor of $C = 0.25 \cdot 10^{-6}$ farad and inductor of $L = 1$ henry. If the initial charge on the capacitor is 10^{-6} coulomb and there is no initial current, find the charge Q as a function of t. Graph the solution and explain the type of oscillations that the graph displays.

5. A mass of 20 kg is oscillating on a spring with the spring constant of 3920 N/m in a medium with the damping constant of 400 kg/sec. If the mass is pulled down additional 2 m and then released, determine the position u as the function of time t. Graph the solution and explain the type of motion that the graph displays.

6. A mass of 0.5 kg stretches a spring .1 m. The mass is acted on by an external force of $\sin \frac{t}{2}$ newtons and moves in a medium that impacts a viscous force with the damping constant of 5 kg/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 0.03 m/sec, determine the position u as the function of time t.

Solutions.

1. (a) The general solution is $u = c_1 \cos t + c_2 \sin t$. With $u(0) = 1$ and $u'(0) = 0$, we have that $u = \cos t$. This is an undamped free oscillator and the solution is a periodic function with a constant amplitude. The first graph on Figure 6 displays u and u'.

   (b) The homogeneous solution is $u_h = c_1 \cos t + c_2 \sin t$. Since $0.8i$ is not the solution of the characteristic equation, the particular solution has the form $u_p = A \cos 0.8t + B \sin 0.8t$. Calculate that $B = 0$ and $A = \frac{25}{18}$. Thus $u = c_1 \cos t + c_2 \sin t + \frac{25}{18} \cos 0.8t$. Using the initial conditions, we have that $c_1 = -\frac{25}{18}$ and $c_2 = 0$. So, $u = \frac{25}{18}(\cos 0.8t - \cos t)$.

   This is an undamped forced oscillator. The second graph on Figure 6 displays u and u'. From the graph, we can see that the mass oscillates with a periodic amplitude. So the oscillations are with beats.

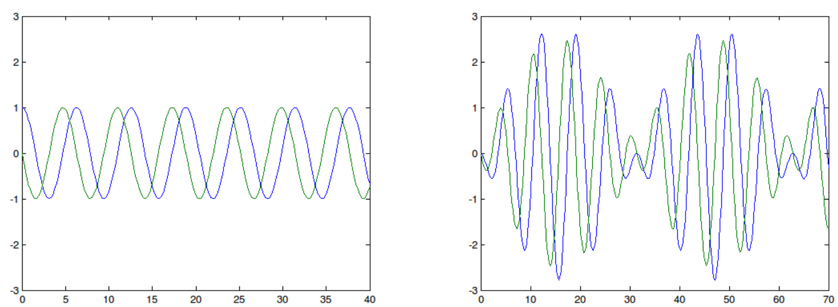

Figure 6: Part (a) Undamped Free Oscillations. Part (b) Undamped Forced Oscillations, Beats

   (c) The homogeneous solution is $u_h = c_1 \cos t + c_2 \sin t$. Since i is the solution of the characteristic equation, the particular solution has the form $u_p = At \cos t + Bt \sin t$. Calculate that $A = 0$ and $B = \frac{1}{4}$. Thus $u = c_1 \cos t + c_2 \sin t + \frac{1}{4} t \sin t$. Using the initial conditions, we have that $c_1 = 0$ and $c_2 = 0$. So, $u = \frac{1}{4} t \sin t$.

   This is an undamped forced oscillator. From the graph (the first graph on Figure 7), we can see that the oscillations have an increasing amplitude. So, the oscillations are with a resonance.

   (d) The solutions of the characteristic equation are $\frac{-1 \pm \sqrt{63}i}{8} = -.125 \pm .992i$. The general solution is $u = c_1 e^{-.125t} \cos .992t + c_2 e^{-.125t} \sin .992t$.. With $u(0) = 1$ and $u'(0) = 0$, we have that $c_1 = 1$ and $c_2 = .126$. $u = e^{-.125t} \cos .992t + .126 e^{-.125t} \sin .992t$.

This is a damped free oscillator. The second graph on Figure 7 displays u and u'. From the graph, we can see that the oscillations have a decreasing amplitude so this is an underdamped oscillator.

(e) The characteristic equation has -1 as a double zero. The general solution is $u = c_1 e^{-t} + c_2 t e^{-t}$. With $u(0) = 1$ and $u'(0) = 0$, we have that $u = e^{-t} + te^{-t} = (1+t)e^{-t}$. The third graph on Figure 7 displays u and u' and we can see that this is an overdamped free case. The mass returns to equilibrium position without oscillations.

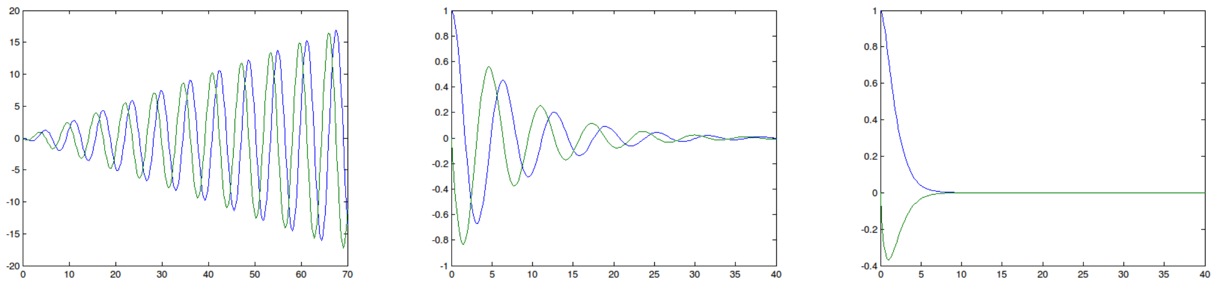

Figure 7: Part (c) Undamped Forced Oscillations, Resonance. Part (d) Damped Free Oscillations, Underdamped case. (e) Damped Free Oscillations, Overdamped case

2. The solutions are not overdamped if the characteristic equation has complex solutions (since just in this case the solution has periodic functions present). The characteristic equation is $r^2 + \gamma r + 9 = 0$. The solutions are $r = \frac{-\gamma \pm \sqrt{\gamma^2 - 36}}{2}$. Thus, the complex solutions are present just if the expression under the root is negative. So, $\gamma^2 - 36 < 0 \Rightarrow (\gamma - 6)(\gamma + 6) < 0$. Since γ is nonnegative, this corresponds to interval $0 \leq \gamma < 6$.

3. Find k first: $k = \frac{mg}{L} = \frac{0.1(9.8)}{0.05} = 19.6$. Since there is no damping and the oscillations are free, the equation of motion is $mu'' + ku = 0$. Thus, $0.1u'' + 19.6u = 0 \Rightarrow u'' + 196u = 0$. The general solution is $u = c_1 \cos 14t + c_2 \sin 14t$. Since the mass is set in motion from the equilibrium position $u(0) = 0$. The initial velocity is 10 cm/sec so $u'(0) = 10$. From the initial conditions, $c_1 = 0$ and $c_2 = \frac{10}{14} = \frac{5}{7}$. Hence, $u = \frac{5}{7} \sin 14t$ cm. These are undamped free oscillations: the solution is a periodic function with a constant amplitude.

   If you leave the initial conditions in centimeters, your solution will be in centimeters too. Alternatively, you can convert to meters. *Careful:* you have to be more careful with units if the motion is damped, forced or both.

   The function $u = \frac{5}{7} \sin 14t$ has first positive zero when $14t = \pi \Rightarrow t = \frac{\pi}{14} = .22$ sec. So, the mass first returns to equilibrium position .22 seconds after it is set in motion.

   The frequency of oscillations is 14 and the period is $\frac{2\pi}{14} = \frac{\pi}{7}$.

4. The circuit equation is $Q'' + 4 \cdot 10^6 Q = 0$. The general solution is $Q = c_1 \cos 2000t + c_2 \sin 2000t$. The initial conditions $Q(0) = 10^{-6}$ and $Q'(0) = 0$ give us that $c_1 = 10^{-6}$ and $c_2 = 0$. Thus $Q = 10^{-6} \cos 2000t$. This is an undamped free oscillator and the solution is a periodic function with a constant amplitude.

5. The equation of motion is $20u'' + 400u' + 3920u = 0$. The roots of characteristic equation $20r^2 + 400r + 3920 = 0$ are $-10 \pm 4\sqrt{6}i$. So, the general solution is $u = e^{-10t}(c_1 \cos 4\sqrt{6}t + c_2 \sin 4\sqrt{6}t)$. The initial condition are $u(0) = 2$ and $u'(0) = 0$. Thus, $c_1 = 2$ and $c_2 = \frac{20}{4\sqrt{6}} = \frac{5}{\sqrt{6}}$ and the solution is $u = e^{-10t}(2\cos 4\sqrt{6}t + \frac{5}{\sqrt{6}} \sin 4\sqrt{6}t)$. This is a damped free oscillator. The graph indicates that the oscillator is underdamped: the oscillations have a decreasing amplitude.

6. $k = \frac{0.5 \cdot 9.8}{.1} = 49$, and $\gamma = 5$ so the equation of motion is $0.5u'' + 5u' + 49u = \sin \frac{t}{2}$. The characteristic equation is $r^2 + 10r + 98 = 0$ and has zeros $-5 \pm \sqrt{73}i$. So, the solution of the homogeneous part is $u_h = c_1 e^{-5t} \cos(\sqrt{73}t) + c_2 e^{-5t} \sin(\sqrt{73}t)$. The particular solution is of the form $u_p = A \cos \frac{t}{2} + B \sin \frac{t}{2}$. You can substitute the derivatives into $u'' + 10u' + 98u = 2 \sin \frac{t}{2}$. Obtain $-\frac{A}{4} + 5B + 98A = 0$ and $-\frac{B}{4} - 5A + 98B = 2 \Rightarrow 391A + 20B = 0$ and $-20A + 391B = 8 \Rightarrow B = \frac{-391A}{20}$ and $(-400 - 391^2)A = 160 \Rightarrow A = -0.001$ and $B = 0.0204$. Thus, $u = c_1 e^{-5t} \cos(\sqrt{73}t) + c_2 e^{-5t} \sin(\sqrt{73}t) - 0.001 \cos \frac{t}{2} + 0.02 \sin \frac{t}{2}$.

   The initial conditions are $u(0) = 0$, $u'(0) = .03$. This gives us that $c_1 = 0.001$ and $-5c_1 + \sqrt{73}c_2 + 0.01 = 0.03 \Rightarrow \sqrt{73}c_2 = 0.025 \Rightarrow c_2 = 0.0029$. Thus, $u = 0.001e^{-5t} \cos(\sqrt{73}t) + 0.0029e^{-5t} \sin(\sqrt{73}t) - 0.001 \cos \frac{t}{2} + 0.02 \sin \frac{t}{2}$.