Differential Equations
Lia Vas

The Second Exam Review

1. **Homogeneous equations with constant coefficients.** Solve the following equations.

   (a) $y'' - 2y' + 5y = 0$ (b) $y''' - 2y'' + y' = 0$

   (c) $y^{(4)} - y = 0$ (d) $y^{(4)} - 5y'' - 36y = 0$

   (e) $y^{(5)} - 32y = 0.$ (f) $y^{(5)} + 32y = 0.$

2. **Non-homogeneous equations with constant coefficients. Variation of Parameters.** Solve the following differential equations. Note that parts (a) and (c) cannot be solved using Undetermined Coefficients method.

   (a) $y'' - 6y' + 9y = x^{-3}e^{3x}.$ (b) $y'' - 5y' + 6y = 2e^{x}$ (c) $y'' + 4y' + 4y = x^{-2}e^{-2x}$

3. **Non-homogeneous equations with constant coefficients. Undetermined Coefficients.** Find general solution of problems (a)–(d). In problems (e)–(g), find the *form* of particular solutions and the general solutions. For (e)–(g), you **do not** have to solve for unknown coefficients in particular solutions.

   (a) $y'' - 5y' + 6y = 4e^{2x}$ (b) $y'' + 4y = 5x^2e^{x}$

   (c) $y'' - 2y' + y = 7xe^{x}$ (d) $y'' + 2y' - 3y = 5\sin 3x$

   (e) $y'' - 3y' - 10y = 3xe^{2x} + 5e^{-2x}$ (f) $y'' - 8y' + 16y = 3x^2 - 5e^{4x}$

   (g) $y'' + 4y' + 13y = -2\sin 3x + e^{-2x}\cos 3x$

4. **Applications of higher order differential equations.** Consider harmonic oscillations described by each of the equations in parts (a)–(c). In each case below, solve the equation, graph the solution and explain the type of motion that the graph displays.

   (a) $u'' + u = \frac{1}{2}\cos 0.8t$, $u(0) = 0$, $u'(0) = 0$;

   (b) $u'' + u = \frac{1}{2}\cos t$, $u(0) = 0$, $u'(0) = 0$;

   (c) $u'' + 2u' + u = 0$, $u(0) = 1$, $u'(0) = 0$;

   (d) A mass of 20 kg is oscillating on a spring with the spring constant of 3920 kg/sec^2 in a medium with the damping constant of 400 kg/sec. If the mass is pulled down additional 2 m and then released, determine the position u as the function of time t. Graph the solution and explain the type of motion that the graph displays.

   (e) A series circuit has capacitor of $C = 0.25 \cdot 10^{-6}$ farad and inductor of $L = 1$ henry. If the initial charge on the capacitor is 10^{-6} coulomb and there is no initial current, find the charge Q as a function of t. Graph the solution and explain the type of oscillations that the graph displays.

   (f) Determine the values of γ for which the equation $u'' + \gamma u' + 9u = 0$ has solutions which are not overdamped.

Solutions

1. Homogeneous Equations.

   (a) $y'' - 2y' + 5y = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2\pm\sqrt{-16}}{2} = 1 \pm 2i \Rightarrow y_1 = e^x \cos 2x$ and $y_2 = e^x \sin 2x$. General solution $y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$.

   (b) $y''' - 2y'' + y' = 0 \Rightarrow r^3 - 2r^2 + r = 0 \Rightarrow r(r-1)^2 = 0 \Rightarrow r = 0$ is a zero and $r = 1$ is a double zero $\Rightarrow y_1 = e^{0x} = 1$, $y_2 = e^x$ and $y_3 = xe^x$. The general solution is $y = c_1 + c_2 e^x + c_3 xe^x$.

   (c) $y^{(4)} - y = 0 \Rightarrow r^4 - 1 = 0 \Rightarrow (r^2 - 1)(r^2 + 1) = 0 \Rightarrow r = \pm 1$ and $r = \pm i \Rightarrow$ the general solution is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

   Alternatively, you can find the four solutions by considering $\sqrt[4]{1} = \sqrt[4]{1e^{0i}} = 1\ e^{\frac{2k\pi}{4} i} = e^{\frac{k\pi}{2} i}$ for $k = 0, 1, 2, 3$. Then $r_0 = 1, r_1 = i, r_2 = -1$ and $r_3 = -i$ yield the same general solution.

   (d) $y^{(4)} - 5y'' - 36y = 0 \Rightarrow r^4 - 5r^2 - 36 = 0 \Rightarrow (r^2 - 9)(r^2 + 4) = 0 \Rightarrow r = \pm 3$ and $r = \pm 2i \Rightarrow$ the general solution is $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 2x + c_4 \sin 2x$.

   (e) $y^{(5)} - 32y = 0 \Rightarrow r^5 - 32 = 0 \Rightarrow r^5 = 32 = 32e^{0i} \Rightarrow r_k = \sqrt[5]{32} e^{\frac{0+2k\pi}{5} i} = 2e^{\frac{2k\pi}{5} i}$ for $k = 0, 1, \ldots, 4$. $r_0 = 2e^{0i} = 2$, $r_1 = 2e^{2\pi i/5} = 2(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}) = 0.618 + 1.902i$, $r_2 = 2e^{4\pi i/5} = 2(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}) = -1.618 + 1.176i$, $r_3 = 2e^{6\pi i/5} = 2(\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}) = -1.618 - 1.176i$, $r_4 = 2e^{8\pi i/5} = 2(\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}) = 0.618 - 1.902i$.

   r_1 and r_4 are conjugated and r_2 and r_3 are conjugated. The general solution is $y = c_1 e^{2x} + c_2 e^{0.618x} \cos 1.902x + c_3 e^{0.618x} \sin 1.902x + c_4 e^{-1.618x} \cos 1.176x + c_5 e^{-1.618x} \sin 1.176x$.

   (f) $y^{(5)} + 32y = 0 \Rightarrow r^5 + 32 = 0 \Rightarrow r^5 = -32 = 32e^{\pi i} \Rightarrow r_k = \sqrt[5]{32} e^{\frac{\pi+2k\pi}{5} i} = 2e^{\frac{(2k+1)\pi}{5} i}$ for $k = 0, 1, \ldots, 4$. $r_0 = 2e^{\pi i/5} = 2(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}) = 1.618 + 1.176i$, $r_1 = 2e^{3\pi i/5} = 2(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}) = -0.618 + 1.902i$, $r_2 = 2e^{\pi i} = 2(\cos \pi + i \sin \pi) = -2$, $r_3 = 2e^{7\pi i/5} = 2(\cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5}) = -0.618 - 1.902i$, $r_4 = 2e^{9\pi i/5} = 2(\cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5}) = 1.618 - 1.176i$.

   r_0 and r_4 are conjugated and r_1 and r_3 are conjugated. The general solution is $y = c_1 e^{-2x} + c_2 e^{1.618x} \cos 1.176x + c_3 e^{1.618x} \sin 1.176x + c_4 e^{-0.618x} \cos 1.902x + c_5 e^{-0.618x} \sin 1.902x$.

2. Variation of Parameters.

   (a) $y_h = c_1 e^{3x} + c_2 xe^{3x}$ and $y_p = \frac{1}{2} x^{-1} e^{3x}$. The general solution is $y = c_1 e^{3x} + c_2 xe^{3x} + \frac{1}{2} x^{-1} e^{3x}$.

   (b) $y_h = c_1 e^{2x} + c_2 e^{3x}$ and $y_p = e^x$. The general solution is $y = c_1 e^{2x} + c_2 e^{3x} + e^x$.

   (c) $y_h = c_1 e^{-2x} + c_2 xe^{-2x}$ and $y_p = -\ln x\ e^{-2x} - e^{-2x}$. The general solution is $y = (c_1 - 1)e^{-2x} + c_2 xe^{-2x} - \ln x\ e^{-2x} = C_1 e^{-2x} + c_2 xe^{-2x} - \ln x\ e^{-2x}$.

3. Undetermined Coefficients.

   (a) $r = 2$ and $r = 3$ are zeros of the characteristic equation and $y_h = c_1 e^{2x} + c_2 e^{3x}$. Then $s = 1$ since 2 is a (single) zero of the characteristic equation, and so $y_p = Axe^{2x}$. Determine A to be -4 so $y_p = -4xe^{2x}$ and the general solution is $y = c_1 e^{2x} + c_2 e^{3x} - 4xe^{2x}$.

   (b) $r = \pm 2i$ are zeros of the characteristic equation and $y_h = c_1 \cos 2x + c_2 \sin 2x$. Then $s = 0$ since 1 is not a zero of the characteristic equation, and so $y_p = x^0(Ax^2 + Bx + C)e^x$. Obtain that $A = 1, B = -\frac{4}{5}$, and $C = -\frac{2}{25}$ so $y_p = (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$ and the general solution is $y = c_1 \cos 2x + c_2 \sin 2x + (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$.

(c) $r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0$ so $r = 1$ is a double zero. So $y_h = c_1e^x + c_2xe^x$. Then $s = 2$ since 1 is double zero of the characteristic equation, and so $y_p = x^2(Ax + B)e^x = (Ax^3 + Bx^2)e^x$. Obtain that $A = \frac{7}{6}$, and $B = 0$ so $y_p = \frac{7}{6}x^3e^x$ and the general solution is $y = c_1e^x + c_2xe^x + \frac{7}{6}x^3e^x$.

(d) $r = -3$ and $r = 1$ are zeros of the characteristic equation and $y_h = c_1e^{-3x} + c_2e^x$. Then $s = 0$ since $3i$ is not a zero of the characteristic equation, and so $y_p = A\cos 3x + B\sin 3x$. Obtain that $A = -\frac{1}{6}$ and $B = -\frac{1}{3}$. So $y_p = -\frac{1}{6}\cos 3x - \frac{1}{3}\sin 3x$. The general solution is $y = c_1e^{-3x} + c_2e^x - \frac{1}{6}\cos 3x - \frac{1}{3}\sin 3x$.

(e) The roots of the characteristic equation $r^2 - 3r - 10 = (r-5)(r+2) = 0$ are 5 and -2 so the homogeneous solution is $y_h = c_1e^{5x} + c_2e^{-2x}$.

You have to consider functions $g_1(x) = 3xe^{2x}$ and $g_2(x) = 5e^{-2x}$ separately and obtain two separate particular solutions y_{p1} and y_{p2}.

For $g_1(x) = 3xe^{2x}$, note that 2 is not a solution of the characteristic equation so $s = 0$. Because of the linear polynomial $3x$ in $g_1(x)$, the first particular solution has the form $y_{p1} = (Ax + B)e^{2x}$.

For $g_2(x) = 5e^{-2x}$, note that -2 is a solution of the characteristic equation so $s = 1$. Because of the constant polynomial 5 in $g_2(x)$, the second particular solution has the form $y_{p1} = x^1Ce^{-2x} = Cxe^{-2x}$.

The general solution has the form $y = c_1e^{5x} + c_2e^{-2x} + (Ax + B)e^{2x} + Cxe^{-2x}$.

(f) The characteristic equation $r^2 - 8r + 16 = (r-4)(r-4) = 0$ has a double zero $r = 4$. The homogeneous solution is $y_h = c_1e^{4x} + c_2xe^{4x}$.

For $g_1(x) = 3x^2 = 3x^2e^{0x}$, note that 0 is not a solution of the characteristic equation so $s = 0$. Because of the quadratic polynomial $3x^2$ in $g_1(x)$, the first particular solution has the form $y_{p1} = Ax^2 + Bx + C$.

For $g_2(x) = -5e^{4x}$, note that 4 is a double solution of the characteristic equation so $s = 2$. Because of the constant polynomial -5 in $g_2(x)$, the second particular solution has the form $y_{p1} = x^2De^{4x} = Dx^2e^{4x}$.

The general solution has the form $y = c_1e^{4x} + c_2xe^{4x} + Ax^2 + Bx + C + Dx^2e^{4x}$.

(g) The characteristic equation $r^2+4r+13 = 0$ has solutions $r = \frac{-4\pm\sqrt{16-52}}{2} = \frac{-4\pm 6i}{2} = -2\pm 3i$. So, the homogeneous solution is $y_h = c_1e^{-2x}\cos 3x + c_2e^{-2x}\sin 3x$.

For $g_1(x) = -2\sin 3x = -2e^{0i}\sin 3x$, note that $0\pm 3i$ are not solutions of the characteristic equation so $s = 0$. Because of the constant polynomial -2 in $g_1(x)$, the first particular solution has the form $y_{p1} = A\cos 3x + B\sin 3x$.

For $g_2(x) = e^{-2x}\cos 3x$, note that $-2 \pm 3i$ is a solution of the characteristic equation so $s = 1$. Because of the constant polynomial 1 in $g_2(x)$, the second particular solution has the form $y_{p1} = x(Ce^{-2x}\cos 3x + De^{-2x}\sin 3x) = Cxe^{-2x}\cos 3x + Dxe^{-2x}\sin 3x$.

The general solution has the form $y = c_1e^{-2x}\cos 3x + c_2e^{-2x}\sin 3x + A\cos 3x + B\sin 3x + Cxe^{-2x}\cos 3x + Dxe^{-2x}\sin 3x$.

4. Applications.

(a) The homogeneous solution is $u_h = c_1\cos t + c_2\sin t$. Since $0.8i$ is not the solution of the characteristic equation, the particular solution has the form $u_p = A\cos 0.8t + B\sin 0.8t$.

Calculate that $B = 0$ and $A = \frac{25}{18}$. Thus $u = c_1 \cos t + c_2 \sin t + \frac{25}{18}\cos 0.8t$. Using the initial conditions, we have that $c_1 = -\frac{25}{18}$ and $c_2 = 0$. So, $u = \frac{25}{18}(\cos 0.8t - \cos t)$. This is an undamped forced oscillator. The graph indicates that the oscillator displays beats.

(b) The homogeneous solution is $u_h = c_1 \cos t + c_2 \sin t$. Since i is the solution of the characteristic equation, the particular solution has the form $u_p = At\cos t + Bt\sin t$. Calculate that $A = 0$ and $B = \frac{1}{4}$. Thus $u = c_1 \cos t + c_2 \sin t + \frac{1}{4}t\sin t$. Using the initial conditions, we have that $c_1 = 0$ and $c_2 = 0$. So, $u = \frac{1}{4}t\sin t$. This is an undamped forced oscillator. The graph indicates that the oscillator displays resonance.

(c) The characteristic equation has -1 as a double zero. The general solution is $u = c_1 e^{-t} + c_2 te^{-t}$. With $u(0) = 1$ and $u'(0) = 0$, we have that $u = e^{-t} + te^{-t} = (1+t)e^{-t}$. Since there are no sine and cosine functions in the general solution the oscillations are overdamped. The graph also supports this conclusion. From the graph, you can also see that the oscillator does not make a single full oscillation.

(d) The equation of motion is $20u'' + 400u' + 3920u = 0$. The roots of characteristic equation $20r^2 + 400r + 3920 = 0$ are $-10 \pm 4\sqrt{6}i$. So, the general solution is $u = e^{-10t}(c_1 \cos 4\sqrt{6}t + c_2 \sin 4\sqrt{6}t)$. The initial condition are $u(0) = 2$ and $u'(0) = 0$. Thus, $c_1 = 2$ and $c_2 = \frac{20}{4\sqrt{6}} = \frac{5}{\sqrt{6}}$ and the solution is $u = e^{-10t}(2\cos 4\sqrt{6}t + \frac{5}{\sqrt{6}}\sin 4\sqrt{6}t)$.
This is a damped free oscillator. The graph indicates that the oscillator is underdamped – the oscillations have decreasing amplitude.

(e) The circuit equation is $Q'' + 4 \cdot 10^6 Q = 0$. The general solution is $Q = c_1 \cos 2000t + c_2 \sin 2000t$. The initial conditions $Q(0) = 10^{-6}$ and $Q'(0) = 0$ give us that $c_1 = 10^{-6}$ and $c_2 = 0$. Thus $Q = 10^{-6}\cos 2000t$. This is an undamped free oscillator and the solution is a periodic function with constant amplitude.

(f) The solutions are not overdamped if the characteristic equation has complex solutions (since just in this case the solution has periodic functions present). The characteristic equation is $r^2 + \gamma r + 9 = 0$. The solutions are $r = \frac{-\gamma \pm \sqrt{\gamma^2 - 36}}{2}$. Thus, the complex solutions are present just if the expression under the root is negative. So, $\gamma^2 - 36 < 0 \Rightarrow (\gamma - 6)(\gamma + 6) < 0$. Since γ is nonnegative, this corresponds to interval $0 \leq \gamma < 6$.