Differential Equations
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The Third Exam Review

1. Use the definition of the Laplace transform to show that $\mathcal{L}[t] = \frac{1}{s^2}$.

2. Find the Laplace transform of the following functions.

   a) $t^4e^{-2t} + \cos 5t - 7$ b) $\int_0^t \tau^3 e^{t-\tau} d\tau$ c) $\int_0^t \sin(2\tau)\cos(2t-2\tau)d\tau$

3. Find the inverse Laplace transform of the following functions.

   a) $\frac{5}{s^2+4}$, b) $\frac{s^2}{(s+1)^3}$, c) $\frac{10}{s^2+3s-4}$, d) $\frac{s+4}{s^2+2s+5}$, e) $\frac{5s^2+3s-2}{s^3+2s^2}$ f) $\frac{3s^2-4s+5}{(s-1)(s^2+1)}$

4. Use the Laplace transform to solve the following initial value problems.

   (a) $y'' - 6y' + 5y = 2$, $y(0) = 0$, $y'(0) = -1$.

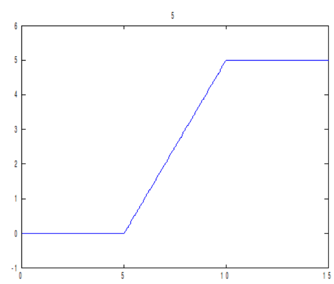
   (b) $y'' + y = \begin{cases} 1, & 5 \leq t < 20, \\ 0, & t < 5 \text{ and } t \geq 20. \end{cases}$ $y(0) = 0$, $y'(0) = 0$. Represent the solution as a piecewise function and graph it on interval [0, 30].

   (c) $y'' + 4y = \delta(t - 4\pi)$, $y(0) = 1$, $y'(0) = 0$.

   (d) $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$. Express your answer in terms of an integral involving function $f(t)$.

5. Assume that an undamped harmonic oscillator is described by the following differential equation.

   $$y'' + 4y = \begin{cases} 0, & t < 5, \\ t-5, & 5 \leq t < 10, \\ 5, & t \geq 10. \end{cases} \qquad y(0) = 0, \; y'(0) = 0.$$

   The function on the right side of the equation is known as **ramp loading** and can be represented as $u_5(t)(t-5) - u_{10}(t)(t-10)$.

   Find the solution, write your answer as a piecewise function, sketch its graph and describe the motion and expand on its meaning.

6. Assume that the initial value problem $y'' + 3y' + 4y = \delta(t - 3)$, $y(0) = 0$, $y'(0) = 0$, models the motion y (in cm) of an oscillator as time t (in seconds) passes. Find the solution, write your answer as a piecewise function, sketch its graph and describe the motion.

7. Solve the integral equation $y(t) + \int_0^t (t-\tau)y(\tau)d\tau = t$.

8. Solve the integro-differential equation $y'(t) + \int_0^t y(t-\tau)e^{-2\tau}d\tau = 1$, $y(0) = 1$.

9. Solve the following system.

   $$\begin{aligned} x' &= -x + y & x(0) &= 1 \\ y' &= -x - y & y(0) &= 2 \end{aligned}$$

Solutions

1. $\mathcal{L}[t] = \int_0^\infty te^{-st}dt$ To evaluate this integral, use the integration by parts with $u = t$ and $dv = e^{-st}$. We have $\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}|_0^\infty$. The limit of the first term for $t \to \infty$ is 0 (you may use L'Hopital's rule to see that). The limit of the second terms is also zero for $t \to \infty$ since $e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$. At $t = 0$, the antiderivative is $\frac{-1}{s^2}$. So, $\mathcal{L}[t] = 0 - \frac{-1}{s^2} = \frac{1}{s^2}$.

2. a) $\frac{24}{(s+2)^5} + \frac{s}{s^2+25} - \frac{7}{s}$ b) The function is the convolution of t^3 and e^t. Thus the Laplace transform is $\mathcal{L}[t^3]\mathcal{L}[e^t] = \frac{6}{s^4}\frac{1}{s-1} = \frac{6}{s^4(s-1)}$. c) The function is the convolution of $\sin 2t$ and $\cos 2t$. Thus the Laplace transform is $\mathcal{L}[\sin 2t]\mathcal{L}[\cos 2t] = \frac{2}{s^2+4}\frac{s}{s^2+4} = \frac{2s}{(s^2+4)^2}$.

3. a) $\mathcal{L}^{-1}[\frac{5}{s^2+4}] = \frac{5}{2}\mathcal{L}^{-1}[\frac{2}{s^2+4}] = \frac{5}{2}\sin 2t$ b) $\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{-2}{(s+1)^2} + \frac{1}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] = e^{-t} - 2te^{-t} + \frac{1}{2}t^2e^{-t}$

   c) Since s^2+3s-4 factors as $(s+4)(s-1)$, in order to find the Laplace transform, we need to find the partial fractions $\frac{A}{s+4} + \frac{B}{s-1}$. We obtain $A = -2$, $B = 2$. So, $\mathcal{L}^{-1}[\frac{-2}{s+4} + \frac{2}{s-1}] = -2e^{-4t} + 2e^t$.

   d) $s^2 + 2s + 5$ cannot be factored in a product of two linear real terms, so you need to write it as sum of squares as $s^2 + 2s + 5 = (s + 2s + 1) + 4 = (s + 1)^2 + 2^2$. Then $\frac{s+4}{s^2+2s+5} = \frac{s+1+3}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2}$. Hence the inverse Laplace is $e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$.

   e) Using the partial fractions, $\frac{5s^2+3s-2}{s^3+2s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{2}{s} - \frac{1}{s^2} + \frac{3}{s+2}$. $\mathcal{L}^{-1}$ is $2 - t + 3e^{-2t}$.

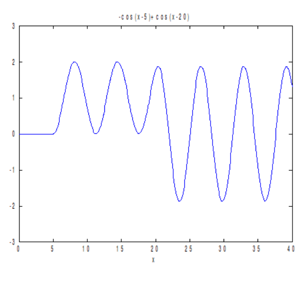
   f) Using the partial fractions, $\frac{3s^2-4s+5}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{2}{s-1} + \frac{s-3}{s^2+1}$ $\mathcal{L}^{-1}$ is $2e^t + \cos t - 3\sin t$.

4. (a) The Laplace transform of the equation is $s^2Y + 1 - 6sY + 5Y = \frac{2}{s} \Rightarrow Y(s^2 - 6s + 5) = \frac{2}{s} - 1 \Rightarrow Y(s-1)(s-5) = \frac{2-s}{s} \Rightarrow Y = \frac{2-s}{s(s-5)(s-1)}$. The partial fraction decomposition is $Y = \frac{2}{5s} - \frac{3}{20(s-5)} - \frac{1}{4(s-1)}$. Thus $y = \frac{2}{5} - \frac{3}{20}e^{5t} - \frac{1}{4}e^t$.

   (b) The function on the right side is a boxcar function given by $u_5(t) - u_{20}(t)$. Taking the Laplace transform of the equation with $\mathcal{L}[y] = Y$, we obtain $s^2Y + Y = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$. From here $Y = (e^{-5s} - e^{-20s})\frac{1}{s(s^2+1)}$. In order to find the inverse Laplace transform, it is sufficient to find the inverse Laplace transform of $\frac{1}{s(s^2+1)}$ and to use the property $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$. So let $F(s) = \frac{1}{s(s^2+1)}$ then $F(s) = \frac{A}{s} + \frac{Bs+C}{s^2+1}$. Find the coefficients to be $A = 1, B = -1$ and $C = 0$ so that $F(s) = \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow f(t) = \mathcal{L}^{-1}[F(s)] = 1 - \cos t$. Thus, the solution is $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[(e^{-5s}F(s) - e^{-20s}F(s)] = u_5(t)f(t-5) - u_{20}(t)f(t-20) =$

$$= u_5(t)(1-\cos(t-5)) - u_{20}(t)(1-\cos(t-20)) = \begin{cases} 0, & t < 5 \\ 1 - \cos(t-5), & 5 \le t < 20, \\ -\cos(t-5) + \cos(t-20), & t \ge 20. \end{cases}$$

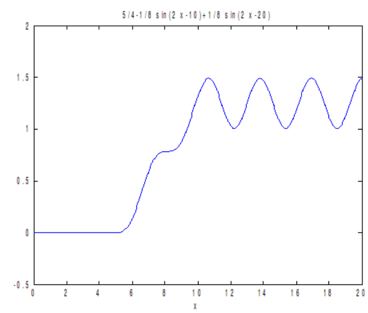
   To graph y using TI83+, you can enter **0(X<5)+(1-cos(X-5))(5≤X<20)+(-cos(X-5)+cos(X-20))(X≥ 20)** as your function. The graph is on the right.



(c) The Laplace transform gives you $s^2Y - s + 4Y = e^{-4\pi s}$. Thus $Y = \frac{e^{-4\pi}+s}{s^2+4}$. Then $y = u_{4\pi}(t)\frac{1}{2}\sin 2(t-4\pi) + \cos 2t = \begin{cases} \cos 2t, & t < 4\pi, \\ \cos 2t + \frac{1}{2}\sin 2(t-4\pi) & t \geq 4\pi. \end{cases}$

(d) Let $Y = \mathcal{L}[y]$ and $F = \mathcal{L}[f]$. The Laplace transform converts the equation to $s^2Y + Y = F$. From here $Y = \frac{F}{s^2+1} = F\frac{1}{s^2+1}$. Applying the inverse Laplace, you obtain $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[F\,\frac{1}{s^2+1}] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[\frac{1}{s^2+1}] = f * \sin t = \int_0^t f(\tau)\sin(t-\tau)d\tau$.

5. The Laplace transform makes the equation into $s^2Y + 4Y = e^{-5s}\frac{1}{s^2} - e^{-10s}\frac{1}{s^2}$. Thus $Y = (e^{-5s} - e^{-10s})\frac{1}{s^2(s^2+4)}$. The fraction $\frac{1}{s^2(s^2+4)}$ decomposes as $\frac{1/4}{s^2} - \frac{1/4}{s^2+4}$ and its inverse Laplace transform is $f(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$. Thus, $y = \mathcal{L}^{-1}[Y] = u_5(t)f(t-5) - u_{10}(t)f(t-10) = u_5(t)(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)) - u_{10}(t)(\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)) =$
$$= \begin{cases} 0, & 0 \leq t < 5, \\ \frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5), & 5 \leq t < 10, \\ \frac{5}{4} - \frac{1}{8}\sin 2(t-5) + \frac{1}{8}\sin 2(t-10), & t \geq 10. \end{cases}$$



The solution is zero before 5. Between 5 and 10, it increases by oscillating about the line $\frac{1}{4}(t-5)$. For $t \geq 10$, the graph becomes one of a simple harmonic oscillation (like shifted sine function) oscillating about 5/4.

6. Let $Y = \mathcal{L}[y]$. Applying the Laplace transform to the equation $y'' + 3y' + 4y = \delta(t-3)$ with $y(0) = y'(0) = 0$ produces $s^2Y + 3sY + 4Y = e^{-3s}$. From here $Y = \frac{e^{-3s}}{s^2+3s+4}$. Complete the denominator of $F(s) = \frac{1}{s^2+3s+4}$ to a sum of squares. $s^2 + 3s + 4 = s^2 + 2s(\frac{3}{2}) + \frac{9}{4} + 4 - \frac{9}{4} = (s+\frac{3}{2})^2 + \frac{7}{4}$. Thus $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{1}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}\mathcal{L}^{-1}[\frac{\frac{\sqrt{7}}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}e^{-3t/2}\sin\frac{\sqrt{7}}{2}t$.
$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{e^{-3s}}{s^2+3s+4}] = \mathcal{L}^{-1}[e^{-3s}F(s)] = u_3(t)\,f(t-3) = u_3(t)\,\frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3) \Rightarrow$
$y = \begin{cases} 0, & t < 3, \\ \frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3), & t \geq 3. \end{cases}$ The object starts oscillating after the first three second. It oscillates with a decreasing amplitude given by $\frac{2}{\sqrt{7}}e^{-3(t-3)/2}$ converging to 0. So the oscillations become negligible in time.

7. The equation is $y + t * y = t$. Taking $\mathcal{L}$, obtain $Y + \frac{1}{s^2}Y = \frac{1}{s^2} \Rightarrow Y = \frac{1}{s^2+1} \Rightarrow y = \sin t$.

8. The equation is $y' + y * e^{-2t} = 1$. Thus $sY - 1 + Y\frac{1}{s+2} = \frac{1}{s} \Rightarrow Y(s + \frac{1}{s+2}) = \frac{1}{s} + 1 \Rightarrow Y\frac{s(s+2)+1}{s+2} = \frac{1+s}{s} \Rightarrow Y = \frac{(1+s)(s+2)}{s(s^2+2s+1)} = \frac{(1+s)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)}$. Find the partial fraction decomposition to be $Y = \frac{2}{s} - \frac{1}{s+1} \Rightarrow y = 2 - e^{-t}$.

9. Let $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$. Taking $\mathcal{L}$ of both equations gives you $sX - 1 = -X + Y$ and $sY - 2 = -X - Y$. From the first equation $Y = sX + X - 1$. Plugging that in the second gives you $s(sX + X - 1) - 2 = -X - (sX + X - 1) \Rightarrow s^2X + 2sX + 2X = s + 3 \Rightarrow X = \frac{s+3}{s^2+2s+2}$. Thus $Y = \frac{s^2+3s+s+3-s^2-2s-2}{s^2+2s+2} = \frac{2s+1}{s^2+2s+2}$.
Then $x = \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}[\frac{s+1+2}{(s+1)^2+1}] = \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1}] = e^{-t}\cos t + 2e^{-t}\sin t$ and $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{2s+1}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{2s+2-1}{(s+1)^2+1}] = \mathcal{L}^{-1}[2\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}] = 2e^{-t}\cos t - e^{-t}\sin t$.