**Mathematical Modeling**
**Lia Vas**

# Dimensional Analysis

One of the key steps in the process of mathematical modeling is to determine the relationship between the variables. Considering the dimensions of those quantities can be useful when determining such relationship.

Dimensional analysis is a method for helping determine how variables are related and for simplifying a mathematical model. Dimensional analysis alone does not give the exact form of an equation, but it can lead to a significant reduction of the number of variables. It is based on two assumptions

1. Physical quantities have dimensions (fundamental are mass $M$, length $L$, and time $T$). Any physical quantity has a dimension which is a product of powers of the basic dimensions $M$, $L$ and $T$.

2. Physical laws are unaltered when changing the units measuring the dimensions.

Units must be taken into consideration when collecting the data as well as when making the list of factors impacting the model and when testing the model. You must check that all the equations in a model are **dimensionally consistent.**

The concept of dimensional consistency (the second assumption above) is related to **dimensional homogenicity**. For example, the equation $t = \sqrt{2s/g}$ that describes the time a body falls from a distance $s$ under gravity is true in all systems, whereas the equation $t = \sqrt{s/16.1}$ is not dimensionally homogeneous because it depends on a particular systems (units of $g$ are neglected so the units of the left and the right side of the equation do not match).

The following table of dimensions of physical entities in the $MLT$ system can be useful.

| | | | |
|---|---|---|---|
| Mass | $M$ | Angular acceleration | $T^{-2}$ |
| Length | $L$ | Momentum | $MLT^{-1}$ |
| Time | $T$ | Angular momentum | $ML^2T^{-1}$ |
| Velocity | $LT^{-1}$ | Density | $ML^{-3}$ |
| Acceleration | $LT^{-2}$ | Viscosity | $ML^{-1}T^{-1}$ |
| Force | $MLT^{-2}$ | Pressure | $ML^{-1}T^{-2}$ |
| Energy, work, heat, torque, entropy | $ML^2T^{-2}$ | Surface tension | $MT^{-2}$ |
| Frequency, angular velocity | $T^{-1}$ | Power | $ML^2T^{-3}$ |

**Example 1.** Find a model describing the terminal velocity of a particle that falls under gravity through a viscous fluid.

When a particle falls through a viscous fluid, the drag force counteracts with the acceleration caused by the gravity and after some time, the velocity stops increasing. This is called the terminal velocity. We can assume that the velocity $v$ depends on the particle's diameter $D$, the viscosity $\mu$ and the acceleration $g$. Also, we will assume that the velocity is directly proportional to the difference between the density $\rho_1$ of the particle and the density $\rho_2$ of the fluid. Thus,

$$v = kD^a\mu^b g^c(\rho_1 - \rho_2)$$

where $k$ is the proportionality (dimensionless) constant and $a, b$ and $c$ are unknown numbers that we will determine using the dimensional analysis.

Note that the density have unit $ML^{-3}$, the gravity has the unit of acceleration $LT^{-2}$ and the viscosity has the unit $ML^{-1}T^{-1}$.

Consider the dimensions of both sides:

$$\begin{aligned} LT^{-1} &= L^a(ML^{-1}T^{-1})^b(LT^{-2})^c(ML^{-3}) \\ &= M^{b+1}L^{a-b+c-3}T^{-b-2c} \end{aligned}$$

Equating the exponents of $M$, $L$ and $T$ on both sides we obtain a system of three equations in three unknowns.

$$\begin{aligned} 0 &= b+1 \\ 1 &= a-b+c-3 \\ -1 &= -b-2c \end{aligned}$$

The solution of the system is $a = 2$, $b = -1$, and $c = 1$. Thus the equation describing the velocity is

$$v = \frac{kD^2g(\rho_1 - \rho_2)}{\mu}.$$

Note that if an expression involving functions as $e^{at}$ or $\sin(at)$ where $t$ is time appears in a model, then $a$ must have dimensions $T^{-1}$.

Also, recall that the dimensions of the derivative are the ration of the dimensions. For example, the derivative of pressure with respect to time has the units

$$\left[\frac{dp}{dt}\right] = \frac{[p]}{[t]} = \frac{ML^{-1}T^{-2}}{T} = ML^{-1}T^{-3}$$

Analogous rule applies to the partial derivatives as well.

**Example 2.** Develop a model which describes the period of a swinging pendulum. Assume that the period $t$ depends on the length $l$, mass $m$, gravitational acceleration $g$, the amplitude $\theta$.

Note that the angle $\theta$ is dimensionless. Thus, from the equation $t = kl^am^bg^c\theta^d$ we obtain a system of just three unknowns $a, b$ and $c$. Solving the system, we obtain that $t = kl^{1/2}g^{-1/2}\theta^d$. To determine the constant $d$, further methods would be needed (e.g. empirical measurements).

**Example 3.** Assume that the air resistance $R$ is impacting the pendulum from Example 2. Develop a model which describes the period of a pendulum.

With $R$ added to the list of the variables, the equation relating them becomes

$$t = kl^am^bg^c\theta^dR^e.$$

The unit consideration yield:

$$T = L^aM^b(LT^{-2})^c(MLT^{-2})^e$$

We obtain the following system:

$$b+e=0, \quad a+c+e=0 \quad \text{and} \quad -2c-2e=1.$$

As there are four variables and just 3 equations, we cannot determine the solution uniquely but solve for three variables in terms of the fourth (free variable). Choose any of the variables to be free. For example, with $b$ a free variable, we obtain $a = 1/2, b = b, c = -1/2 + b, e = -b$. So, $t = kl^{1/2}m^b g^{-1/2+b}\theta^d R^{-b}$. Now we can group the variables into groups of dimensionless products and obtain the following model:

$$t = k\left(\frac{l}{g}\right)^{1/2}\left(\frac{mg}{R}\right)^b \theta^d.$$

Note that solving the system of equations for $c$ instead of $b$ would give us $a = 1/2, b = 1/2 + c, c = c, e = -1/2 - c$. This gives another valid model $t = kl^{1/2}m^{1/2+c}g^c\theta^d R^{-1/2-c} = k\left(\frac{lm}{R}\right)^{1/2}\left(\frac{mg}{R}\right)^c \theta^d$.

The example above illustrate that the model may not be completely determined. The two free variables in the previous example could further be determined using empirical modeling. In cases when the number of free parameters is even larger a method of dimensional analysis known as **Buckingham's Theorem** [1] is especially handy.

The idea of this method is to group the variables into dimensionless products $\Pi_1, \Pi_2, \ldots, \Pi_n$ and obtain the solution in the form $f(\Pi_1, \Pi_2, \ldots, \Pi_n) = 0$ or, equivalently if solving for one of the products (say the first one) $\Pi_1 = g(\Pi_2, \ldots, \Pi_n)$. In case that there is just one product $\Pi_1$, this last equation becomes $\Pi_1 =$ constant.

For example, the dimensionless products in Example 2 are $\Pi_1 = \frac{t^2 g}{l}$, and $\Pi_2 = \theta$, and in Example 3 are $\Pi_1 = \frac{t^2 g}{l}$, $\Pi_2 = \theta$, and $\Pi_3 = \frac{mg}{R}$.

To obtain the products $\Pi_1, \Pi_2, \ldots, \Pi_n$, you can follow the steps below.

1. List all the variables and their units.

2. Make the exponents of variables some unknown quantities $(a, b, c \ldots)$.

3. Write a linear system of three equations (one for each of $M$, $L$ and $T$) in unknowns $a, b, c \ldots$

4. Find the general solution of the system. Note that if you have more than three unknowns, you will not be able to find an unique solution. In this case, group the variables so that each have the same free parameter in the exponent. This gives you the products $\Pi_1, \Pi_2, \ldots, \Pi_n$. In case you need to solve for particular quantity, **choose the parameter in the exponent of that quantity for one of the free parameters.**

5. The desired model has the form $\Pi_1 = k\Pi_2^{a_2} \ldots \Pi_n^{a_n}$ where $k$ is proportionality constant and parameters $a_2, \ldots, a_n$ are free variables among $a, b, c \ldots$. If there is just one product $\Pi_1$, the model is $\Pi_1 = k$.

**Example 4.** Find a model determining the terminal velocity of a raindrop falling from a motionless cloud.

[1] The exact statement of Buckingham's Theorem is as follows: An equation is dimensionally homogeneous if and only if it can be put into the form $f(\Pi_1, \Pi_2, \ldots, \Pi_n) = 0$ where $f$ is a function and $(\Pi_1, \Pi_2, \ldots, \Pi_n)$ is a complete system of dimensionless products.

The variables are: size of raindrop $r$ (we can assume that the raindrop is a sphere), density $\rho$, the air viscosity $\mu$, and the velocity $v$. Any dimensionless product between these variables must be of the form $v^a r^b g^c \rho^d \mu^e$ with dimensions

$$(LT^{-1})^a L^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$$

This gives us a system of three equations and five unknowns.

$$\begin{aligned} d+e &= 0 \\ a+b+c-3d-e &= 0 \\ -a-2c-e &= 0 \end{aligned}$$

Thus, there will be two free variables. Since you need to solve for velocity in the end, and the exponent of velocity is $a$, choose $a$ to be one of your free variables. Choose any other parameter to be the other free variable. For example, choosing $e$ for another free variable, we obtain the general solution

$$a = a, \quad b = -\frac{1}{2}a - \frac{3}{2}e, \quad c = -\frac{1}{2}a - \frac{1}{2}e, \quad d = -e, \quad e = e.$$

Substituting that solution back in $v^a r^b g^c \rho^d \mu^e$ we obtain

$$v^a r^{-\frac{1}{2}a-\frac{3}{2}e} g^{-\frac{1}{2}a-\frac{1}{2}e} \rho^{-e} \mu^e = \left(vr^{-1/2}g^{-1/2}\right)^a \left(r^{-3/2}g^{-1/2}\rho^{-1}\mu\right)^e$$

This gives us two dimensionless product $\Pi_1 = vr^{-1/2}g^{-1/2} = \frac{v}{\sqrt{rg}}$ and $\Pi_2 = r^{-3/2}g^{-1/2}\rho^{-1}\mu = \frac{\mu}{\sqrt{r^3 g \rho^2}}$.

The model $\Pi_1 = g(\Pi_2)$ in this case boils down to $\Pi_1 = k\Pi_2^e$ producing the equation

$$\frac{v}{\sqrt{rg}} = k\left(\frac{\mu}{\sqrt{r^3 g\rho^2}}\right)^e \quad \Rightarrow v = k\sqrt{rg}\left(\frac{\mu}{\sqrt{r^3 g\rho^2}}\right)^e$$

where $k$ is a proportionality constant.

**Example 5.** Find a model for Example 3 using Buckingam's theorem.

Start with $l^a m^b g^c \theta^d R^e t^f$. Considering the units, you obtain $L^a M^b (LT^{-2})^c (MLT^{-2})^e T^f$. The three equations are

$$a + c + e = 0, \quad b + e = 0, \quad \text{and} \quad -2c - 2e + f = 0.$$

Chose $f$ to be one of your free variables. $d$ is also a free variable in this problem. When choosing $e$ for the remaining free variable, and solving for $a$, $b$ and $c$, one obtains that

$$a = -\frac{1}{2}f, \quad b = -e, \quad \text{and} \quad c = -e + \frac{1}{2}f$$

Substituting back in $l^a m^b g^c \theta^d R^e t^f$, one obtains

$$l^{-\frac{1}{2}f} m^{-e} g^{-e+\frac{1}{2}f} \theta^d R^e t^f = \theta^d \left(m^{-1}g^{-1}R\right)^e \left(l^{-\frac{1}{2}} g^{\frac{1}{2}} t\right)^f$$

This yields three dimensionless products $\Pi_1 = \theta$, $\Pi_2 = m^{-1}g^{-1}R = \frac{R}{mg}$ and $\Pi_3 = l^{-\frac{1}{2}}g^{\frac{1}{2}}t = \frac{\sqrt{g}t}{\sqrt{l}}$.

A feasible model can be obtained in the form $\Pi_3 = k\Pi_1^d\Pi_2^e$. Solving for $t$, you obtain

$$t = k\left(\frac{l}{g}\right)^{1/2}\theta^d\left(\frac{R}{mg}\right)^e.$$

**Practice Problems**

1. Suppose that you are driving a van down a highway with gusty winds. How does the speed of your vehicle affect the wind force you are experiencing. Hint: relate the variables for area, force, density of the wind and velocity.

2. Find how the centrifugal force $F$ of a particle depends on its mass $m$, velocity $v$ and the radius $r$ of the curvature of its path.

3. Find the volume flow rate $\frac{dV}{dt}$ of blood flowing in an artery as a function of the pressure $P$ drop per unit length of artery, the radius $r$, the blood density $\rho$ and the blood viscosity $\mu$.

4. The speed of sound $v$ in a gas depends on the pressure $p$ and the density $\rho$. Find how the speed $v$ depends on $p$ and $\rho$.

**Solutions**

1. $F = kv^2A\rho$ where $k$ is the proportionality constant, $A$ the area of the van, $F$ the force, $\rho$ the density of the wind and $v$ the velocity of the van.

2. $F = \frac{kmv^2}{r}$ where $k$ is the proportionality constant.

3. Using Buckingham's Theorem you should get two dimensionless products. One possible solution: $\Pi_1 = \frac{dV}{dt}\frac{1}{Pr^3\mu}$ and $\Pi_2 = \frac{Pr^2\rho}{\mu^2}$. In this case $\frac{dV}{dt} = kPr^3\mu\left(\frac{Pr^2\rho}{\mu^2}\right)^e$. Alternatively, starting from $\frac{dV}{dt} = kP^ar^b\rho^c\mu^d$, you can get the same answer.

4. $v = k\frac{\sqrt{p}}{\sqrt{\rho}}$ where $k$ is the proportionality constant.