Special Topics Course
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Cycle Graphs of Groups

The whole idea here is that sometimes it is better to represent a group graphically that using presentations, listing all the elements or using Cayley tables. This is true especially for groups with larger number of elements when Cayley table become hard to read.

So, the idea is to represent a group graphically as a graph. A **graph** is a set of objects called **vertices** or nodes some of which are connected by links called **edges**. So, graph is a set of vertices together with edges between them.

Graph are used to represent different objects and the relations between them. The following paragraph on applications of graph is taken from Wikipedia: Many applications of graph theory exist in the form of network analysis. These split broadly into two categories. Firstly, analysis to determine structural properties of a network, such as whether or not it is a scale-free network, or a small-world network. Secondly, analysis to find a measurable quantity within the network, for example, for a transportation network, the level of vehicular flow within any portion of it. Graph theory is also used to study molecules in science. In condensed matter physics, the three dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. For example, Franzblau's shortest-path (SP) rings. Source: http://en.wikipedia.org/wiki/Graph_theory

Cycle graph is a graph that corresponds to a group. Group elements are vertices. Two elements a and b are a part of a cycle if $b = a^n$ for some n. Note that this has a consequence that a cycle graph of a cyclic group is a cycle (as every elements is of the form a^n). On Wikipedia site, 1 is always represented by a dark circle and the rest of the group elements by light circles.

Let us look at some examples. For all the following examples, look at pictures on wikipedia site **http://en.wikipedia.org/wiki/List_of_small_groups**

Scroll down the page till the second table titled "Combined List". Let us look at groups there.

The trivial group with only one element 1 will be represented by a single vertex, no edges.

The group with two elements C_2 will be represented by two vertices connected with one line.

The group with 3 elements C_3 will be represented as a triangle.

The cyclic group with 4 elements C_4 will be represented by a square. The other group with 4 elements $C_2 \times C_2$ (not isomorphic to C_4, explain why) This group is generated by 2 elements a and b. Its elements are $1, a, b$ and ab. None of these elements are in a cycle as a^n is never b nor ab and same goes for b^n or $(ab)^n$ for any n. As squares of all the 3 nontrivial elements are 1, this group is represented as 3 vertices all connected just with the fourth central vertex (see picture on wikipedia site).

C_5 is a cycle with 5 vertices.

There are 2 nonisomorphic groups of order 6: $C_6 = C_3 \times C_2$ (explain why equality here) and D_3. C_6 is a cycle with 6 vertices. D_3 has the following elements: identity a and a^2 in one cycle and b, ab, a^2b - all with square 1. So, its cycle graph will be a triangle (corresponding to $1, a, a^2$) and 3 edges coming out from (with b, ab, a^2b as vertices on the other ends). See picture on wikipedia site.

Similarly, any dihedral group D_n will have a cycle with n vertices (corresponding to elements $1, a, a^2, \ldots a^{n-1}$) and n edges sticking out of 1 for remaining symmetries $a, ab, a^2b, \ldots a^{n-1}b$. Illustrate this with examples for D_4, D_5 etc. You could also go over examples for $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, $C_3 \times C_3$ and that will probably fill all the time you have.

If you need further input, let me know.