Math Methods 1
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Review for Exam 1

1. **Surface Integrals**.

   (a) Find the area of the surface $z = y^2 + x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Write down the parametric equations of the paraboloid and use them to find the surface area.

   (b) Find the area of the cone $z = \sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.

   (c) Evaluate $\int\int_S xz\, dS$ where S is the part of the plane $4x + 2y + z = 8$ that lies in the first octant.

   (d) Evaluate $\int\int_S yz\, dS$ where S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

   (e) Evaluate $\int\int_S z\, dS$ where S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

2. **Flux integral.** Find the flux integrals of the given vector fields over the specified surfaces.

   (a) $\mathbf{f} = (y, x, z)$ over the part of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$.

   (b) $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

   (c) $\mathbf{f} = (xze^y, -xze^y, z)$ over the part of the plane $x + y + z = 1$ in the first octant with the upward orientation.

   (d) $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm1, \pm1, \pm1)$.

3. **Line integrals** with and without using Stokes Theorem.

   (a) Find the work done by the force field $\mathbf{f} = (-y, x, x^2 + y^2)$ when a particle moves under its influence along the positively oriented boundary of the part of the paraboloid $z = 4-x^2-y^2$ in the first octant. (i) Without using Stokes' Theorem; (ii) Using Stokes' Theorem.

   (b) Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for $\mathbf{f} = (-y^2, x, z^2)$ and the curve C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ oriented upwards. (i) Without using Stokes' Theorem; (ii) Using Stokes' Theorem.

   (c) Find the work done by the force field $\mathbf{f} = (x + z^2, y + x^2, z + y^2)$ when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant oriented upwards.

4. **Flux integral using the Divergence Theorem.**

   (a) Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm1, \pm1, \pm1)$.

   (b) Find the flux of the vector field $\mathbf{f} = (z, y, x)$ over the unit sphere.

(c) Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

(d) Find the flux of the vector field $\mathbf{f} = (ye^z, 2y, xe^y)$ over the boundary of the region enclosed by the cylinder $x^2 + y^2 = 9$, $z = 0$ and $z = 4 - y$.

Solutions.

More detailed solutions can be found on the class handout.

1. (a) $\mathbf{r} = (r\cos t, r\sin t, r^2) \Rightarrow \mathbf{r}_r = (\cos t, \sin t, 2r)$ and $\mathbf{r}_t = (-r\sin t, r\cos t, 0) \Rightarrow \mathbf{r}_r \times \mathbf{r}_t = (-2r^2\cos t, -2r^2\sin t, r) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{4r^4\cos^2 t + 4r^4\sin^2 t + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1}$. The projection in the xy-plane is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Thus $0 \le t \le 2\pi$ and $1 \le r \le 2$. So, $S = \int_0^{2\pi} dt \int_1^2 r\sqrt{4r^2+1}dr = 2\pi 4.91 = 30.85$.

   (b) $\mathbf{r} = (r\cos t, r\sin t, r) \Rightarrow \mathbf{r}_r = (\cos t, \sin t, 1)$ and $\mathbf{r}_t = (-r\sin t, r\cos t, 0) \Rightarrow \mathbf{r}_r \times \mathbf{r}_t = (-r\cos t,\ -r\sin t, r) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{r^2\cos^2 t + r^2\sin^2 t + r^2} = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}r$. The bounds are $0 \le t \le 2\pi$ and $2 \le r \le 3$. So, $S = \int_0^{2\pi} dt \int_2^3 \sqrt{2}r dr = 2\pi\sqrt{2}(\frac{9}{2} - \frac{4}{2}) = 5\pi\sqrt{2}$.

   (c) $\mathbf{r} = (x, y, 8 - 4x - 2y) \Rightarrow \mathbf{r}_x = (1, 0, -4), \mathbf{r}_y = (0, 1, -2) \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = (4, 2, 1) \Rightarrow dS = \sqrt{16+4+1}dxdy = \sqrt{21}dxdy$. The bounds are $0 \le x \le 2$, $0 \le y \le 4 - 2x$. $\int\int_S xz\, dS = \int_0^2 \int_0^{4-2x} x(8-4x-2y)\sqrt{21}dxdy = \sqrt{21}\int_0^2 (8xy - 4x^2y - xy^2)|_0^{4-2x} dx = \text{(simplify)} = \sqrt{21}\int_0^2 (16x - 16x^2 + 4x^3)dx = \frac{16\sqrt{21}}{3}$.

   (d) $\mathbf{r} = (x, y, y+3) \Rightarrow dS = \sqrt{2}dxdy$. $\int\int y(y+3)\sqrt{2}dxdy$ = (using polar coordinates) $= \int_0^{2\pi}\int_0^1 r\sin t(r\sin t + 3)\ \sqrt{2}rdrdt = \sqrt{2}\int_0^{2\pi} \sin t(\frac{1}{4}\sin t + 1) = \sqrt{2}\frac{\pi}{4}$. Alternatively, $\mathbf{r} = (r\cos t, r\sin t, r\sin t + 3) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{2}r$. The integral is $\int_0^{2\pi}\int_0^1 r\sin t(r\sin t + 3)r\sqrt{2}\ drdt = \sqrt{2}\frac{\pi}{4}$.

   (e) $\mathbf{r} = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi) \Rightarrow dS = 4\sin\phi d\theta d\phi$. The bounds are $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{2}$ (we need to stay above the xy-plane) so the integral is $\int\int_S zdS = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\phi\ 4\sin\phi\ d\theta d\phi = 2\pi\int_0^{\pi/2} 2\cos\phi\ 4\sin\phi\ d\phi = 2\pi\ 8 \cdot \frac{1}{2} = 8\pi$.

2. (a) $\mathbf{r} = (r\cos\theta, r\sin\theta, 1 - r^2) \Rightarrow d\mathbf{S} = (2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta$ and $\mathbf{f}\cdot d\mathbf{S} = (4r^3\sin\theta\cos\theta + r - r^3)drd\theta$. The bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. The integral is $\int\int_S \mathbf{f}d\mathbf{S} = \int_0^{2\pi}\int_0^1 (4r^3\sin\theta\cos\theta + r - r^3)drd\theta = \int_0^{2\pi}(\sin\theta\cos\theta + \frac{1}{2} - \frac{1}{4})d\theta = \frac{\pi}{2}$.

   (b) The flux integral is the sum of the integrals over the paraboloid and over the plane $z = 0$. The first integral is $\frac{\pi}{2}$ by the previous problem. The second integral is $\int\int_S (y, x, 0)\cdot(0, 0, -1)dxdy = \int\int 0dxdy = 0$. So, the total is $\frac{\pi}{2}$.

   (c) $\mathbf{r} = (x, y, 1 - x - y) \Rightarrow d\mathbf{S} = (1, 1, 1)dxdy$. $\int\int_S (xze^y, -xze^y, 1 - x - y)\cdot(1, 1, 1)dxdy = \int_0^1\int_0^{1-x}(1 - x - y)dxdy = \int_0^1 (1 - x - x(1-x) - \frac{1}{2}(1-x)^2) = \frac{1}{6}$.

   (d) The cube consists of 6 sides. On top and bottom $z = \pm 1$ and $-1 \le x, y \le 1$ so $\int\int_S (x, 2y, \pm 3)\cdot(0, 0, \pm 1)dxdy = \int_{-1}^1\int_{-1}^1 3dxdy = 12$. On the left and right $y = \pm 1$ and $-1 \le x, z \le 1$ so $\int\int_S (x, \pm 2, 3z)\cdot(0, \pm 1, 0)dxdz = \int_{-1}^1\int_{-1}^1 2dxdz = 8$. On the front and back $x = \pm 1$ and $-1 \le y, z \le 1$ so $\int\int_S (\pm 1, 2y, 3z)\cdot(\pm 1, 0, 0)dydz = \int_{-1}^1\int_{-1}^1 1dydz = 4$. Thus, the total flux is 2(12+8+4)=48.

3. (a) Without Stokes: The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant consists of three curves, C_1 in xy-plane, C_2 in yz-plane, and C_3 in xz-plane.

C_1 C_1 in is the xy-plane $z=0 \Rightarrow 0=4-x^2-y^2 \Rightarrow x^2+y^2=4$ which has parametric equations $x=2\cos t, y=2\sin t, z=0$. The bounds are $0 \le t \le \frac{\pi}{2}$. Thus, $dx=-2\sin t dt, dy = 2\cos t dt, dz=0$. and $\int_{C_1} -ydx+xdy+(x^2+y^2)dz = \int_0^{\pi/2}(4\sin^2 t+4\cos^2 t+4(0))dt = \int_0^{\pi/2} 4dt = 2\pi$.

C_2 C_2 is in the yz-plane $x=0 \Rightarrow z=4-0^2-y^2 \Rightarrow z=4-y^2 \Rightarrow x=0, y=y, z=4-y^2 \Rightarrow dx=0, dy=dy, dz=-2ydy$. The bounds are 2 to 0 and $\int_{C_2} -ydx+xdy+(x^2+y^2)dz = \int_2^0(0+0+(0+y^2)(-2y)dy = \int_2^0 -2y^3dy = \frac{-y^4}{4}\Big|_2^0 = 8$.

C_3 C_3 is in the xz-plane $y=0 \Rightarrow z=4-x^2-0^2 \Rightarrow z=4-x^2 \Rightarrow x=x, y=0, z=4-x^2 \Rightarrow dx=dx, dy=0, dz=-2xdx$. The bounds are 0 to 2 and $\int_{C_3} -ydx+xdy+(x^2+y^2)dz = \int_0^2(0+0+(x^2+0^2)(-2x)dx = \int_0^2 -2x^3dx = \frac{-x^4}{4}\Big|_0^2 = -8$

The total work is the sum of work done along C_1, C_2, and C_3. Hence $W = \int_{C_1}+\int_{C_2}+\int_{C_3} \mathbf{f}\cdot d\mathbf{r} = 2\pi+8-8=2\pi$.

With Stokes: $\text{curl}\mathbf{f} = (2y,-2x,2)$. The paraboloid can be parametrized by $\mathbf{r}=(r\cos\theta, r\sin\theta, 4-r^2) \Rightarrow d\mathbf{S} = (2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta$. The bounds are $0\le\theta\le\frac{\pi}{2}$ and $0 \le r \le 2$. $\text{curl}\mathbf{f}d\mathbf{S} = 2rdrd\theta$. Thus,

$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{f}d\mathbf{S} = \int_0^{\pi/2}\int_0^2 2rdrd\theta = 2\frac{\pi}{2}\frac{r^2}{2}\Big|_0^2 = 2\frac{\pi}{2}(2) = 2\pi.$$

(b) Without Stokes: C has parametrization $x=\cos t$, $y=\sin t$, $z=2-\sin y$, $0\le t\le 2\pi$. $\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C -y^2dx+xdy+z^2dz = \int_0^{2\pi}\sin^3 tdt+\cos^2 tdt+(2-\sin t)^2\cos tdt = \pi$.

With Stokes: $\text{curl}\mathbf{f} = (0,0,1+2y)$. $\mathbf{r}=(x,y,2-y) \Rightarrow d\mathbf{S}=(0,1,1)dxdy$. $\int\int_S \text{curl}\mathbf{f}d\mathbf{S} = \int\int_S(1+2y)dxdy = \int_0^{2\pi}\int_0^1(1+2r\sin t)rdrdt = \int_0^{2\pi}(\frac{1}{2}+\frac{2}{3}\sin t)dt = \pi$.

(c) When using Stokes' Theorem: $\text{curl}\mathbf{f}=(2y,2z,2x)$. $\mathbf{r}=(2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi) \Rightarrow d\mathbf{S} = (4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi)d\phi d\theta$. $\int\int_S \text{curl}\mathbf{f}d\mathbf{S} = 16\int_0^{\pi/2}\int_0^{\pi/2}(\sin^3\phi\cos\theta\sin\theta + \sin^2\phi\sin\theta\cos\phi+\sin^2\phi\cos\phi\cos\theta)d\phi d\theta = 16\int_0^{\pi/2}(\frac{1}{2}\sin^3\phi+2\sin^2\phi\cos\phi)d\phi d\theta = 16$.

4. (a) It is **much** easier to evaluate the integral using the Divergence Theorem because otherwise you have to do *six* flux integrals. $\text{div}\mathbf{f}=1+2+3=6$ and so $\int\int_S \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 6dxdydz = \int_{-1}^1\int_{-1}^1\int_{-1}^1 6dxdydz = 6x|_{-1}^1 y|_{-1}^1 z|_{-1}^1 = 6(2)^3 = 48$.

(b) It is easier to evaluate the integral using the Divergence Theorem than directly because finding the flux integral directly involves long computation of $d\mathbf{S}$. $\text{div}\mathbf{f}=1$. Thus, $\int\int_S \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 1dxdydz = \int_0^{2\pi}\int_0^\pi\int_0^1 r^2\sin\phi drd\phi d\theta = 2\pi(1+1)\frac{1}{3} = \frac{4\pi}{3}$.

(c) $\text{div}\mathbf{f}=1$. Thus, $\int\int_S \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 1dxdydz = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} rdzdrd\theta = \int_0^{2\pi}\int_0^1(1-r^2)rdrd\theta = 2\pi(\frac{1}{2}-\frac{1}{4}) = \frac{\pi}{2}$.

(d) Without the Divergence Theorem, one would have to evaluate *three* flux integrals and with the Divergence Theorem, just one triple integral. Calculate that $\text{div}\mathbf{f}=2$. Thus, $\int\int_S \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 2dxdydz$. Using cylindrical coordinates, the bounds are $0\le\theta\le 2\pi$, $0\le r\le 3$ and $0 \le z \le 4-y = 4-r\sin\theta$. The integral is $\int_0^{2\pi}\int_0^3\int_0^{4-r\sin\theta} 2\,rdrd\theta dz = \int_0^{2\pi}\int_0^3 2r(4-r\sin\theta)drd\theta = \int_0^{2\pi}\int_0^3(8r-2r^2\sin\theta)drd\theta = \int_0^{2\pi}(36-18\sin\theta)d\theta = 36\theta+18\cos\theta|_0^{2\pi} = 72\pi$.