Math Methods 1
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Review for Exam 1

1. **Surface Integrals**.

   (a) Find the area of the surface $z = y^2 + x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Write down the parametric equations of the paraboloid and use them to find the surface area.

   (b) Find the area of the cone $z = \sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.

   (c) Evaluate $\int\int_S yz\, dS$, surface S is the part of the plane $x + y + z = 1$ that lies in the first octant.

   (d) Evaluate $\int\int_S yz\, dS$, surface S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

   (e) Find the mass of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ if it has constant density $\rho = a$.

2. **Flux integral.** Find the flux integrals of the given vector fields over the specified surfaces.

   (a) $\mathbf{f} = (y, x, z)$ over the part of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$.

   (b) $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

   (c) $\mathbf{f} = (xze^y, -xze^y, z)$ over the part of the plane $x + y + z = 1$ in the first octant with the upward orientation.

   (d) $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

3. **Line integrals** with and without using Stokes Theorem.

   (a) Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for $\mathbf{f} = (x + y^2, y + z^2, z + x^2)$ and the curve C is the intersection of the plane $x + y + z = 1$ and the coordinate planes. (i) Without using Stokes' Theorem; (ii) Using Stokes' Theorem.

   (b) Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for $\mathbf{f} = (-y^2, x, z^2)$ and the curve C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ oriented upwards. (i) Without using Stokes' Theorem; (ii) Using Stokes' Theorem.

   (c) Show that the total work done by the force field $\mathbf{f} = (yz, xz, xy)$ moving the particle along the intersection of the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$ above the xy-plane is $\int_C \mathbf{f} \cdot d\mathbf{r} = 0$. When using Stokes' Theorem, this problem becomes much shorter then without using it.

   (d) Find the work done by the force field $\mathbf{f} = (x + z^2, y + x^2, z + y^2)$ when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant oriented upwards.

4. **Flux integral using the Divergence Theorem.**

(a) Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

(b) Find the flux of the vector field $\mathbf{f} = (z, y, x)$ over the unit sphere.

(c) Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

(d) Find the flux of the vector field $\mathbf{f} = (xy, yz, xz)$ over the boundary of the region enclosed by the cylinder $x^2 + y^2 = 1$, $z = 0$ and $z = 1$.

(e) Find the flux of the vector field $\mathbf{f} = (ye^z, y^2, xe^y)$ over the boundary of the region enclosed by the cylinder $x^2 + y^2 = 9$, $z = 0$ and $z = y - 3$.

(f) Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm 1, \pm 1, \pm 1)$ *without the top.*

Solutions. More detailed solutions can be found on the class handout.

1. (a) The paraboloid can be parametrized by $\mathbf{r} = (r\cos t, r\sin t, r^2) \Rightarrow \mathbf{r}_r = (x_r, y_r, z_r) = (\cos t, \sin t, 2r)$ and $\mathbf{r}_t = (x_t, y_t, z_t) = (-r\sin t, r\cos t, 0) \Rightarrow \mathbf{r}_r \times \mathbf{r}_t = (-2r^2\cos t, -2r^2\sin t, r) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{4r^4\cos^2 t + 4r^4\sin^2 t + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1}$. The projection in the xy-plane is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Thus $0 \le t \le 2\pi$ and $1 \le r \le 2$. So, $S = \int_0^{2\pi} dt \int_1^2 r\sqrt{4r^2+1}dr = 2\pi 4.91 = 30.85$.

   (b) The cone can be parametrized by $\mathbf{r} = (r\cos t, r\sin t, r) \Rightarrow \mathbf{r}_r = (\cos t, \sin t, 1)$ and $\mathbf{r}_t = (-r\sin t, r\cos t, 0) \Rightarrow \mathbf{r}_r \times \mathbf{r}_t = (-r\cos t, -r\sin t, r) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{r^2\cos^2 t + r^2\sin^2 t + r^2} = \sqrt{r^2+r^2} = \sqrt{2r^2} = \sqrt{2}r$. The bounds are $0 \le t \le 2\pi$ and $2 \le r \le 3$. So, $S = \int_0^{2\pi} dt \int_2^3 \sqrt{2}rdr = 2\pi\sqrt{2}(\frac{9}{2} - \frac{4}{2}) = 5\pi\sqrt{2}$.

   (c) $z = 1 - x - y \Rightarrow z_x = -1, z_y = -1$. $dS = \sqrt{1+1+1}dxdy = \sqrt{3}dxdy$. Bounds $0 \le x \le 1$, $0 \le y \le 1 - x$. $\int\int_S yz\ dS = \int_0^1\int_0^{1-x} y(1-x-y)\sqrt{3}dxdy = \sqrt{3}\int_0^1((1-x)\frac{y^2}{2} - \frac{y^3}{3})|_0^{1-x}dx = \sqrt{3}\int_0^1 \frac{(1-x)^3}{6}dx = \frac{\sqrt{3}}{24}$.

   (d) With $\mathbf{r} = (x, y, \sqrt{x^2+y^2}+2)$ $dS = \sqrt{1+1}dxdy = \sqrt{2}dxdy$. $\int\int y(y+3)\sqrt{2}dxdy = \int_0^{2\pi}\int_0^1 r\sin t(r\sin t+3)\sqrt{2}rdrdt = \sqrt{2}\int_0^{2\pi}\sin t(\frac{1}{4}\sin t+1) = \sqrt{2}\frac{\pi}{4}$. Alternatively, $\mathbf{r} = (r\cos t, r\sin t, r\sin t + 3) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{2}r$. The integral is $\int_0^{2\pi}\int_0^1 r\sin t(r\sin t+3)r\sqrt{2}\ drdt = \sqrt{2}\frac{\pi}{4}$.

   (e) On the sphere $x = 2\cos\theta\sin\phi$, $y = 2\sin\theta\sin\phi$ and $z = 2\cos\phi$. Calculate $|\mathbf{r}_\theta \times \mathbf{r}_\phi|$ to be $4\sin\phi$. Thus $m = \int\int_S adS = a\int_0^{2\pi}\int_0^{\pi/2} 4\sin\phi\ d\theta d\phi = 8a\pi$.

2. (a) $\int\int_S (y, x, 1-x^2-y^2)\cdot(2x, 2y, 1)dxdy = \int\int_S(2xy + 2xy + 1 - x^2 - y^2)dxdy$. In polar coordinates, we have $\int_0^{2\pi}\int_0^1(4r^2\sin t\cos t + 1 - r^2)rdrdt = \int_0^{2\pi}(\sin t\cos t + \frac{1}{4})dt = \frac{\pi}{2}$.

   (b) The flux integral is the sum of the integrals over the paraboloid and over the plane $z = 0$. The first integral is $\frac{\pi}{2}$ by the previous problem. The second integral is $\int\int_S(y, x, 0)\cdot(0, 0, -1)dxdy = \int\int 0dxdy = 0$. So, the total is $\frac{\pi}{2}$.

   (c) $\int\int_S(xze^y, -xze^y, 1-x-y)\cdot(1,1,1)dxdy = \int_0^1\int_0^{1-x}(1-x-y)dxdy = \int_0^1(1-x-x(1-x) - \frac{1}{2}(1-x)^2) = \frac{1}{6}$.

   (d) The cube consists of 6 sides. On top and bottom $z = \pm 1$ and $-1 \le x, y \le 1$ so $\int\int_S(x, 2y, \pm 3)\cdot(0,0,\pm 1)dxdy = \int_{-1}^1\int_{-1}^1 3dxdy = 12$. On the left and right $y = \pm 1$ and

$-1 \le x, z \le 1$ so $\int\int_S (x, \pm 2, 3z) \cdot (0, \pm 1, 0) dxdz = \int_{-1}^{1} \int_{-1}^{1} 2dxdz = 8$. On the front and back $x = \pm 1$ and $-1 \le y, z \le 1$ so $\int\int_S (\pm 1, 2y, 3z) \cdot (\pm 1, 0, 0) dydz = \int_{-1}^{1} \int_{-1}^{1} 1dydz = 4$. Thus, the total flux is 2(12+8+4)=48.

3. (a) Without Stokes: The curve C consists of three parts C_1, C_2 and C_3 which are in the intersection of the plane and (1) the plane $z = 6$, (2) xz-plane, and (3) yz-plane, respectively.

   On $C_1 : x = x$, $y = 1 - x$ and $z = 0 \Rightarrow dx = dx$, $dy = -dx$ and $dz = 0$ and the bounds are from 1 to 0. $\int_{C_1} \mathbf{f} d\mathbf{r} = \int_{C_1} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_1^0 (x + (1-x)^2)dx + (1-x)(-1)dx = \int_1^0 (x + 1 - 2x + x^2 - 1 + x)dx = \int_1^0 x^2 dx = \frac{-1}{3}$.

   On $C_2 : x = 0$, $y = y$, $z = 1 - y \Rightarrow dx = 0$, $dy = dy$ and $dz = -dy$. The bounds are from 1 to 0. $\int_{C_2} \mathbf{f} d\mathbf{r} = \int_{C_2} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_1^0 (y + (1-y)^2)dy + (1-y)(-1)dy = \int_1^0 (y + 1 - 2y + y^2 - 1 + y) = \int_1^0 y^2 dy = \frac{-1}{3}$.

   On $C_3 : x = x$, $y = 0$, $z = 1 - x \Rightarrow dx = dx$, $dy = 0$ and $dz = -dx$. The bounds are from 0 to 1. $\int_{C_3} \mathbf{f} d\mathbf{r} = \int_{C_3} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_0^1 xdx + (1 - x + x^2)(-1)dx = \int_0^1 (2x - 1 - x^2)dx = 1 - 1 - \frac{1}{3} = \frac{-1}{3}$. Thus $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{-1}{3} - \frac{1}{3} - \frac{1}{3} = -1$.

   With Stokes: Calculate that curl$\mathbf{f} = (-2z, -2x, -2y)$. On the plane $\mathbf{r} = (x, y, 1 - x - y)$ $d\mathbf{S} = (1, 0, -1) \times (0, 1, -1)dxdy = (1, 1, 1)dxdy$. Thus $\int\int_S \text{curl}\mathbf{f} d\mathbf{S} = \int_0^1 \int_0^{1-x} (-2z - 2x - 2y)\, dxdy = \int_0^1 \int_0^{1-x} (-2 + 2x + 2y - 2x - 2y)\, dxdy = \int_0^1 -2(1-x)dx = -2(1 - \frac{1}{2}) = -1$.

   (b) Without Stokes: C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - \sin y$, $0 \le t \le 2\pi$. $\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C -y^2 dx + xdy + z^2 dz = \int_0^{2\pi} \sin^3 t dt + \cos^2 t dt + (2 - \sin t)^2 \cos t dt = \pi$.

   With Stokes: curl$\mathbf{f} = (0, 0, 1 + 2y)$. $d\mathbf{S} = (0, 1, 1)dxdy$. $\int\int_S \text{curl}\mathbf{f} d\mathbf{S} = \int\int_S (1 + 2y)dxdy = \int_0^{2\pi} \int_0^1 (1 + 2r \sin t) r dr dt = \int_0^{2\pi} (\frac{1}{2} + \frac{2}{3} \sin t)dt = \pi$.

   (c) curl$\mathbf{f} = 0$. Thus $0 = \int\int_S \text{curl}\mathbf{f} d\mathbf{S} = \int_C \mathbf{f} \cdot d\mathbf{r}$.

   (d) When using Stokes' Theorem: curl$\mathbf{f} = (2y, 2z, 2x)$. $\mathbf{r} = (2\cos\theta \sin\phi, 2\sin\theta \sin\phi, 2\cos\phi) \Rightarrow d\mathbf{S} = (4\sin^2\phi \cos\theta, 4\sin^2\phi \sin\theta, 4\sin\phi \cos\phi)d\phi d\theta$. $\int\int_S \text{curl}\mathbf{f} d\mathbf{S} = 16 \int_0^{\pi/2} \int_0^{\pi/2} (\sin^3\phi \cos\theta \sin\theta + \sin^2\phi \sin\theta \cos\phi + \sin^2\phi \cos\phi \cos\theta)d\phi d\theta = 16 \int_0^{\pi/2} (\frac{1}{2} \sin^3\phi + 2\sin^2\phi \cos\phi)d\phi d\theta = 16$.

4. (a) div$\mathbf{f} = 1+2+3 = 6$ and so $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 6dxdydz = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} 6dxdydz = 6(2)^3 = 48$.

   (b) It is easier to use the divergence theorem. div$\mathbf{f} = 1$. Thus, $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 1dxdydz = \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 \sin\phi dr d\phi d\theta = 2\pi(1 + 1)\frac{1}{3} = \frac{4\pi}{3}$.

   (c) div$\mathbf{f} = 1$. Thus, $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 1dxdydz = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r dz dr d\theta = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta = 2\pi(\frac{1}{2} - \frac{1}{4}) = \frac{\pi}{2}$.

   (d) div$\mathbf{f} = y + z + x$. Thus, $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V (x + y + z)dxdydz = \int_0^{2\pi} \int_0^1 \int_0^1 (r\cos t + r\sin t + z) r dr d\theta dz = \int_0^{2\pi} \int_0^1 (r\cos t + r\sin t + \frac{1}{2}) r dr d\theta = \int_0^{2\pi} (\frac{1}{3}\cos t + \frac{1}{3}\sin t + \frac{1}{4})d\theta = \frac{\pi}{2}$.

   (e) Use the divergence theorem (much easier than finding flux directly). div$\mathbf{f} = 2y$. Thus, $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 2ydxdydz = \int\int 2yz|_{y-3}^{0} dxdy = \int\int 2y(-y + 3)dxdy$. With polar coordinates, $0 \le t \le 2\pi$ and $0 \le r \le 3$. The integral is $\int_0^{2\pi} \int_0^3 (-2r^3 \sin^2 t + 6r^2 \sin t)drdt = \int_0^{2\pi} (-\frac{81}{2} \sin^2 t + 54 \sin t)dt = -\frac{81\pi}{2}$

   (f) div$\mathbf{f} = 1 + 2 + 3 = 6$. To use the Divergence Theorem, the top has to be considered. $\int\int_{\text{top}} (x, 2y, 3) \cdot (0, 0, 1)dxdy = \int_{-1}^{1} \int_{-1}^{1} 3dxdy = 12$. Since $\int\int_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} + \int\int_{\text{top}} \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 6dxdydz$, we have that $\int\int_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 6dxdydz - \int\int_{\text{top}} \mathbf{f} \cdot d\mathbf{S} = 48 - 12 = 36$.