Math Methods 1
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Review for Exam 2

1. **Analytic Functions. Complex integrals. Cauchy's Integral Formula.**

   (a) Check if the following functions are analytic.
   (i) $f(z) = ze^{5iz^2}$, (ii) $f(x+iy) = x^2y + ixy^2$, (iii) $f(x+iy) = x^2 - y^2 + 2xyi$.

   (b) Check if analytic functions with real part equal to the given functions u exist. If so, find all analytic functions that have real parts equal to u.
   (i) $u = xe^{3y}$ (ii) $u = x^2 + 3x - y^2 + 5y$.

   (c) Evaluate $\int \text{Re}(z)\, dz$ over the line segment $x = 1$ from (1,0) to (1,1) and the line segment $y = 1$ from (1,1) to (0,1).

   (d) Evaluate $\int z^4 dz$ (i) over the upper-half of the unit circle traversed counterclockwise, (ii) over the unit circle traversed counterclockwise.

   (e) Let $f(z) = z^3 - 2z + e^{z-2}$ and let C be the circle of radius 3 in xy-plane. Evaluate
   (i) $\int_C f(z)dz$ and (ii) $\int_C \frac{f(z)}{z-2} dz$.

   (f) Evaluate $\oint_C \frac{1}{z} dz$ over the square C with sides (1,0), (0,1), (-1,0), (0,-1).

2. **Laurent Series**. Find the power series expansions of the functions $f(z)$ centered at indicated point a. Then, determine all the singularities of $f(z)$, classify their types and find the residues.

   (a) $f(z) = \frac{z}{1-z^2}, \quad z = 0$

   (b) $f(z) = \frac{e^{z-1}}{(z-1)^2}, \quad z = 1$

   (c) $f(z) = \frac{1-\cos z}{z^2}, \quad z = 0$

   (d) $f(z) = z \cos \frac{1}{z}, \quad z = 0$

3. **Complex integrals with Residue Theorem.** Use the Residue Theorem to evaluate the complex integrals of given function $f(z)$ over the given contour C. Express your answers in $a + bi$ form.

   (a) $f(z) = \frac{1}{z^2(z^2+1)}$, C is the circle of radius 1 centered at $\frac{i}{2}$.

   (b) $f(z) = \frac{e^z}{z^2-4}$, C is the circle of radius 5 centered at the origin.

   (c) $f(z) = \frac{1}{(z-2)^2(4+z)}$, C is the circle of radius 3 centered at the origin.

   (d) $f(z) = \frac{1}{(z-2)^2(4+z)}$, C is the circle of radius 5 centered at the origin.

   (e) $f(z) = \frac{e^z}{(z-1)^5}$, C is the boundary of the right half of the disc with radius 2.

   (f) $f(z) = z \cos \frac{1}{z}$, C is the square with vertices (1,0), (0,1), (-1, 0) and (0,-1).

   (g) $f(z) = \frac{1}{(z^2-2z+2)^2}$, C is the boundary of the upper half of the disc with radius R where $R > 2$.

Solutions. More detailed solutions can be found on the class handout.

1. (a) (i) $f(z)$ has derivative $f'(z) = e^{5iz^2} + 10iz^2e^{5iz^2}$ which is continuous. Thus, f is analytic.

(ii) For $f(x+iy) = x^2y + ixy^2$, $u = x^2y$ and $v = xy^2$. Check the Cauchy-Riemann equations. $\frac{\partial u}{\partial x} = 2xy = \frac{\partial v}{\partial y}$ but $\frac{\partial u}{\partial y} = x^2$ and $\frac{\partial v}{\partial x} = y^2$ so the second equation fails. Thus, f is not analytic.

(iii) For $f(x+iy) = x^2 - y^2 + 2xyi$, $u = x^2 + y^2$ and $v = 2xy$. Check if the Cauchy-Riemann equations hold. $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = \frac{\partial v}{\partial x}$ so both equations hold. Thus, f is analytic.

(b) (i) Check if u satisfies Laplace equation. $\frac{\partial u}{\partial x} = e^{3y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial u}{\partial y} = 3xe^{3y} \Rightarrow \frac{\partial^2 u}{\partial y^2} = 9xe^{3y}$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 - 9xe^{3y} \neq 0$, no such analytic function exists. (ii) Check the Laplace equation: $\frac{\partial u}{\partial x} = 2x + 3 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$ and $\frac{\partial u}{\partial y} = -2y + 5 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$.

Since $2x + 3 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ you can find v as $v = \int (2x+3)dy = 2xy + 3y + g(x)$ and you can find g from the second Cauchy-Riemann equation $2y - 5 = \frac{-\partial u}{\partial y} = \frac{\partial v}{\partial x} = 2y + g'(x) \Rightarrow g'(x) = -5 \Rightarrow$ $g(x) = -5x + c$. Thus, $v = 2xy + 3y - 5x + c$. In this case $f(z) = x^2 + 3x - y^2 + 5y + i(2xy + 3y - 5x + c) = x^2 + 2ixy + y^2 + 3(x+iy) + 5(-ix+y) = (x+iy)^2 + 3(x+iy) - 5i(x+iy) = z^2 + 3z - 5iz$.

(c) On the first line segment $x = 1$, $y = y$ and $0 \leq y \leq 1$. $z = 1 + iy$ so Rez=1 and $dz = 0dx + idy = idy$. $\int$ Re $(z)dz = \int_0^1 idy = iy|_0^1 = i$. On the second line segment $x = x$, $y = 1$ and the x-values are decreasing from 1 to 0. $z = x + i$ and Re $z = x$, $dz = dx + 0i = dx$. $\int$ Re $(z)dz = \int_1^0 xdx = \frac{x^2}{2}|_1^0 = -\frac{1}{2}$. So, the total integral is $i - \frac{1}{2}$.

(d) (i) On the unit circle $x = \cos t$, and $y = \sin t$, so $z = \cos t + i \sin t = e^{it}$. Thus, $z^4 = e^{4it}$ and $dz = e^{it}idt$. The bounds for t are $0 \leq t \leq \pi$ so $\int z^4 dz = \int_0^\pi e^{4it}e^{it}idt = i\int_0^\pi e^{5it}dt =$ $\frac{1}{5}(e^{5\pi i} - e^{0i}) = \frac{1}{5}(\cos 5\pi + i \sin 5\pi - 1) = \frac{-2}{5}$. (ii) $f(z) = z^4$ is analytic ($f'(z) = 4z^3$ is continuous). Thus, the integral is zero by Cauchy's Theorem.

(e) $f(z) = z^3 - 2z + e^{z-2}$ is analytic (derivative $3z^2 - 2 + e^{z-2}$ is continuous) so the integral in part (i) is zero by Cauchy's Theorem. By Cauchy's integral formula integral in part (ii) is equal to $2\pi i f(2) = 2\pi i(8 - 4 + 1) = 10\pi i$.

(f) With $f(z) = 1$ and $a = 0$ the Cauchy's integral formula gives you $2\pi i f(0) = 2\pi i$.

2. (a) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} \Rightarrow f(z) = \frac{z}{1-z^2} = \sum_{n=0}^{\infty} z^{2n+1}$. The only singularities of $f(z)$ are ± 1 and they are poles of order 1. The residues are $\lim_{z\to 1}(z-1)\frac{z}{(1-z)(1+z)} =$ $\lim_{z\to 1} \frac{z}{-(1+z)} = -\frac{1}{2}$ and $\lim_{z\to -1}(z+1)\frac{z}{(1-z)(1+z)} = \lim_{z\to -1} \frac{z}{1-z} = \frac{-1}{2}$.

(b) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \Rightarrow \frac{e^{z-1}}{(z-1)^2} = \frac{1}{(z-1)^2}\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(z-1)^{n-2}}{n!} =$ $\frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \ldots$. The only singularity is $z = 1$ and it is a pole of the order 2. The coefficient with the term with $\frac{1}{z-1}$ is 1 so the residue is 1.

(c) $f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2} - \frac{1}{z^2}\cos z = \frac{1}{z^2} - \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z^2} - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-2}}{(2n)!} = \frac{1}{z^2} - \frac{1}{z^2} +$ $\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots$ Thus, there are no terms with negative exponents. So, the only singularity $z = 0$ is a removable singularity.

(d) $f(z) = z \cos \frac{1}{z} = z\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n-1}} = z - \frac{1}{2! z} + \frac{1}{4! z^3} - \ldots$. The only singularity is 0 and it is an essential singularity (infinitely many terms with z in denominator). From the power series expansion, the coefficient with $\frac{1}{z}$ is $\frac{-1}{2}$ so that is the residue.

3. (a) Since $z^2+1=(z-i)(z+i)$, f has three singularities $0, i$ and $-i$. 0 is a pole of order 2 and $\pm i$ are poles of the first order. Just 0 and i are inside curve C. The residue at 0 is $\lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right)=\lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{z^2+1}\right)=\lim_{z\to 0}\frac{-2z}{(z^2+1)^2}=0$. The residue at i is $\lim_{z\to i}(z-i)f(z)=\lim_{z\to i}\frac{1}{z^2(z+i)}=\frac{1}{-1(i+i)}=\frac{-1}{2i}=\frac{i}{2}$. Using the Residue Theorem, the integral is equal to $2\pi i(0+\frac{i}{2})=-\pi$.

(b) $\frac{e^z}{z^2-4}=\frac{e^z}{(z-2)(z+2)}$ has two poles of order 1, 2 and -2 and both are in C. The residue at 2 is $\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}\frac{e^z}{(z+2)}=\frac{e^2}{4}$. The residue at -2 is $\lim_{z\to -2}(z+2)f(z)=\lim_{z\to 2}\frac{e^z}{(z-2)}=\frac{-e^{-2}}{4}$. Thus, the integral is $2\pi i(\frac{e^2}{4}-\frac{e^{-2}}{4})=\frac{\pi i}{2}(e^2-e^{-2})$.

(c) $f(z)=\frac{1}{(z-2)^2(4+z)}$ has two singularities: 2 is a pole of order 2 and -4 is a pole of order 1. 2 is in C and -4 is not.

The residue at 2 is $\lim_{z\to 2}\frac{d}{dz}\left((z-2)^2 f(z)\right)=\lim_{z\to 2}\frac{d}{dz}\left(\frac{1}{4+z}\right)=\lim_{z\to 2}\frac{-1}{(4+z)^2}=\frac{-1}{36}$ so the integral is $\frac{-\pi i}{18}$.

(d) Find the residue at -4 for function from part (c). It is $\lim_{z\to -4}(z+4)f(z)=\lim_{z\to -4}\frac{1}{(z-2)^2}=\frac{1}{36}$. So, the integral is the product of $2\pi i(\frac{-1}{36}+\frac{1}{36})=0$.

(e) $f(z)=\frac{e^z}{(z-1)^5}$. The only singularity is 1 and it is a pole of order 5. The residue at 1 is $\frac{1}{4!}\lim_{z\to 1}\frac{d^4}{dz^4}\left((z-1)^5 f(z)\right)=\frac{1}{24}\lim_{z\to 1}\frac{d^4}{dz^4}\left(e^z\right)=\frac{1}{24}\lim_{z\to 1}e^z=\frac{e^1}{24}=\frac{e}{24}$ so the integral is $2\pi i\frac{e}{24}=\frac{e\pi i}{12}$.

(f) Using problem 1 (d), the residue at 0 is $\frac{-1}{2}$ and so the integral is equal to $2\pi i\frac{-1}{2}=-\pi i$.

(g) The zeros of the denominator of $f(z)=\frac{1}{(z^2-2z+2)^2}$ are $z=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}=1\pm i$. Thus $f(z)=\frac{1}{(z-(i+1))^2(z-(1-i))^2}$. So, there are two poles the second order $1\pm i$ and just $1+i$ is inside C. The residue at $1+i$ is $\lim_{z\to 1+i}\frac{d}{dz}\left((z-(1+i))^2 f(z)\right)=\lim_{z\to 1+i}\frac{d}{dz}\left(\frac{1}{(z-(1-i))^2}\right)=\lim_{z\to 1+i}\frac{-2}{(z-1+i)^3}=\frac{-2}{(2i)^3}=\frac{-2}{8(-i)}=\frac{1}{4i}=\frac{-i}{4}$. Thus the integral is equal to $2\pi i\frac{-i}{4}=\frac{\pi}{2}$.