Math Methods 1
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Review for Exam 3

1. **Fourier Series.**

   (a) The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the boxcar function $f(x) = \begin{cases} 1 & 0 \le x < \pi \\ -1 & -\pi \le x < 0 \end{cases}$. Find its Fourier series expansion and use it to find the sum of series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

   (b) Find the Fourier cosine series for $f(x) = x^2$ for $0 < x \le 2$. Then use the Fourier series expansion and Parseval's Theorem to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

   (c) Find the Fourier cosine and Fourier sine expansion of $f(x) = \begin{cases} x & 0 < x \le 1 \\ 2 - x & 1 < x < 2 \end{cases}$. Use the Fourier sine expansion to find the sum of series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

   (d) The output from an electronic oscillator is the sawtooth function $f(t) = t$ for $0 \le t \le 1$ that keeps repeating with period 1. Sketch this function and find its complex Fourier series. Using this series and Parseval's Theorem, find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

2. **Fourier Transform.**

   (a) Find the Fourier and the inverse Fourier transforms of the boxcar function $f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$ Express your answer as real functions.

   (b) Find the Fourier and the Fourier cosine transforms of $f(t) = e^{-t}$, $t > 0$, $f(t) = 0$ otherwise.

   (c) Find cosine Fourier transform of $f(t) = 2t - 3$ for $0 < t < 3/2$, $f(x) = 0$ otherwise.

3. **Series Solutions. Regular point.**

   (a) Consider the equation $(1 - x)^2 y'' - 2y = 0$. Show that $x = 0$ is a regular point of this equation. Then find the series solution at $x = 0$. Write your solution in the closed form and determine the radius of the convergence of the solution.

   (b) Consider Hermite equation $y'' - 2xy' + 4y = 0$. Show that $x = 0$ is a regular point of this equation. Find the series solutions of the given equation about $x = 0$. Find the closed form of one solution and list first few terms of the second solution. Determine the interval of convergence.

Solutions.

1. Fourier Series. More detailed solutions can be found on the class handout.

   (a) Graph the function and note it is odd. Thus, $a_n = 0$. Since $L = \pi$ $(T = 2\pi)$, the coefficients b_n can be computed as $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} \sin nx dx = \frac{-2}{n\pi} \cos nx|_0^{\pi} = \frac{-2}{n\pi}((-1)^n - 1)$. Note that $b_{2k} = 0$ and $b_{2k+1} = \frac{-2}{n\pi}(-2) = \frac{4}{(2k+1)\pi}$. Hence, $f(x) = \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} \sin nx = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1} = \frac{4}{\pi}(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots)$.

(b) Note that x^2 is already an even function. So, consider this function on interval [-2, 2] and replicate its graph on this domain outside of this interval too to create a periodic function of period $T = 4$ (thus $L = 2$). Since the new function is even too, $b_n = 0$.

$a_n = \int_0^2 x^2 \cos\frac{n\pi x}{2}dx$. Using integration by parts twice, obtain that $a_n = \frac{2}{n\pi}x^2 \sin\frac{n\pi x}{2}|_0^2 - \frac{4}{n\pi}\int_0^2 x \sin\frac{n\pi x}{2}dx = \frac{8}{n^2\pi^2}x\cos\frac{n\pi x}{2}|_0^2 - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}|_0^2 = \frac{16}{n^2\pi^2}\cos n\pi = \frac{16(-1)^n}{n^2\pi^2}$. Note that this formula works just for $n > 0$ so a_0 has to be computed separately $a_0 = \int_0^2 x^2 dx = \frac{x^3}{3}|_0^2 = \frac{8}{3}$. Thus, the Fourier series is $x^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$.

By Parseval's Theorem, $\frac{1}{2}\int_0^2 x^4 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{16}{9} + \frac{16^2}{2\pi^4}\sum_{n=1}^{\infty}\left(\frac{1}{n^4} + 0\right)$. Note that integral on the left side is $\frac{1}{2}\int_0^2 x^4 dx = \frac{16}{5}$. Dividing the equation above by 16 produces $\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

(c) **Cosine expansion.** First extend the function symmetrically with respect to y-axis so that it is defined on basic period [-2,2] and that it is *even*. Thus $T = 4$ and $L = 2$. The coefficients b_n are zero in this case and the coefficients a_n can be computed as follows. $a_n = \int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^1 x\cos\frac{n\pi x}{2}dx + \int_1^2(2-x)\cos\frac{n\pi x}{2}dx = \left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)|_0^1 + \left(\frac{2(2-x)}{n\pi}\sin\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)|_1^2 = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi = \frac{4}{n^2\pi^2}(2\cos\frac{n\pi}{2} - 1 - \cos n\pi)$. If $n = 2k+1$ is odd, $a_n = \frac{4}{(2k+1)^2\pi^2}(0 - 1 + 1) = 0$. If $n = 2k$ is even, $a_n = \frac{4}{(2k)^2\pi^2}(2(-1)^k - 1 - 1)$. Because of the part with $(-1)^k$, we can distinguish two more cases depending on whether k is even or odd. Thus, if $k = 2l$ is even, $a_n = \frac{4}{(4l)^2\pi^2}(2-1-1) = 0$. If $k = 2l+1$ is odd, $a_n = \frac{4}{(2(2l+1))^2\pi^2}(2(-1) - 1 - 1) = \frac{-16}{(4l+2)^2\pi^2} = \frac{-4}{(2l+1)^2\pi^2}$. If $n = 0$, $a_0 = \int_0^2 f(x)dx = \int_0^1 xdx + \int_1^2(2-x)dx = \frac{1}{2} + \frac{1}{2} = 1$. So, $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos\frac{(4l+2)\pi x}{2} = \frac{1}{2} - \frac{4}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos(2l+1)\pi x$.

Sine expansion. First extend the function symmetrically with respect to the origin so that it is defined on basic period [-2,2] and that it is *odd*. Thus $T = 4$ and $L = 2$. The coefficients a_n are zero in this case and the coefficients b_n can be computed as follows. $b_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2(2-x)\sin\frac{n\pi x}{2}dx = (\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2})|_0^1 + \left(\frac{-2(2-x)}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)|_1^2 = \underline{\frac{-2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \underline{\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$. This is 0 if n is even. If $n = 2k+1$, this is $\frac{8(-1)^k}{(2k+1)^2\pi^2}$. So, $f(x) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2}$.

To find the sum of $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, note that when $x = 1$ the function $f(1)$ is equal to 1 and its Fourier sine expansion is equal to $\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi}{2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}(-1)^n = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. So $\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 1 \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

(d) $T = 1$, $f(t) = \sum_{n=-\infty}^{\infty}c_n e^{2n\pi it}$ and $c_n = \int_0^1 te^{-2n\pi it} = \frac{t}{-2n\pi i}e^{-2n\pi it}|_0^1 + \frac{1}{4n^2\pi^2}e^{-2n\pi it}|_0^1 = \frac{1}{-2n\pi i}e^{-2n\pi i} + \frac{1}{4n^2\pi^2}e^{-2n\pi i} - \frac{1}{4n^2\pi^2}$. Note that $e^{-2n\pi i} = \cos(-2n\pi) + i\sin(-2n\pi) = 1$. Thus $c_n = \frac{1}{-2n\pi i} + 0 = \frac{i}{2n\pi}$. Note that $c_{-n} = \frac{-i}{2n\pi} = \overline{c_n}$. $c_0 = \int_0^1 t\ dt = \frac{1}{2}$. This gives us $f(t) = \frac{1}{2} + \sum_{n=-\infty, n\neq 0}^{\infty}\frac{i}{2n\pi}e^{2n\pi it}$. Note also that $a_0 = 2c_0 = 1$, $a_n = 0$ for $n > 0$ and $b_n = \frac{-1}{n\pi}$. By Parseval's Theorem, $\int_0^1 x^2 dx = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} \Rightarrow \frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

2. Fourier Transform.

(a) $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$. Since $f(t) = 0$ for $t < -1$ and $t > 1$, and $f(t) = 1$ for $-1 \leq t \leq 1$, we have that $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-i\omega t}dt = \frac{-1}{\sqrt{2\pi}i\omega}e^{-i\omega t}|_{-1}^{1} = \frac{-1}{\sqrt{2\pi}\omega}\frac{e^{-i\omega}-e^{i\omega}}{i} = \frac{1}{\sqrt{2\pi}\omega}\frac{e^{i\omega}-e^{-i\omega}}{i} = \frac{2}{\sqrt{2\pi}\omega}\frac{e^{i\omega}-e^{-i\omega}}{2i} = \frac{2}{\sqrt{2\pi}\omega}\sin\omega = \frac{2}{\sqrt{2\pi}}\text{sinc}\omega$.

The inverse transform of the boxcar function can be obtained as $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{i\omega t}d\omega = \frac{1}{\sqrt{2\pi}it}(e^{it} - e^{-it}) = \frac{2}{\sqrt{2\pi}t}\frac{e^{it}-e^{-it}}{2i} = \frac{2}{\sqrt{2\pi}t}\sin t = \frac{2}{\sqrt{2\pi}}\frac{\sin t}{t} = \frac{2}{\sqrt{2\pi}}\text{sinc}t$.

(b) Fourier transform: $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-t}e^{-i\omega t}dt = \frac{-1}{\sqrt{2\pi}(1+i\omega)}e^{-(1+i\omega)t}|_0^{\infty} = \frac{1}{\sqrt{2\pi}(1+i\omega)}$.

Fourier Cosine transform: $F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-t}\cos\omega t\ dt$. Using two integration by parts with $u = e^{-t}$, we have that $F(\omega) = \sqrt{\frac{2}{\pi}}(\frac{1}{\omega}e^{-t}\sin\omega t|_0^{\infty} + \frac{1}{\omega}\int_0^{\infty} e^{-t}\sin\omega t\ dt) = \sqrt{\frac{2}{\pi}}(0 - \frac{1}{\omega^2}e^{-t}\cos\omega t|_0^{\infty} - \frac{1}{\omega^2}\int_0^{\infty} e^{-t}\cos\omega t\ dt) = \sqrt{\frac{2}{\pi}}(\frac{1}{\omega^2} - \frac{1}{\omega^2}\sqrt{\frac{\pi}{2}}F(\omega)$. Solving for $F(\omega)$ gives your $F(\omega)(1 + \frac{1}{\omega^2}) = \sqrt{\frac{2}{\pi}}\frac{1}{\omega^2}$. Multiply by ω^2 to get $F(\omega)(\omega^2+1) = \sqrt{\frac{2}{\pi}} \Rightarrow F(\omega) = \sqrt{\frac{2}{\pi}}\ \frac{1}{\omega^2+1}$.

(c) $F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{3/2}(2t-3)\cos\omega t\ \ dt = \sqrt{\frac{2}{\pi}}(\frac{2t-3}{\omega}\sin\omega t|_0^{3/2} - \frac{2}{\omega}\int_0^{3/2}\sin\omega t\ \ dt) = \sqrt{\frac{2}{\pi}}(0 + \frac{2}{\omega^2}\cos\omega t|_0^{3/2}) = \sqrt{\frac{2}{\pi}}\frac{2}{\omega^2}(\cos\frac{3\omega}{2} - 1) = \frac{2\sqrt{2}}{\omega^2\sqrt{\pi}}(\cos\frac{3\omega}{2} - 1)$.

3. Series solutions. More detailed solutions can be found on the class handout.

(a) Dividing by $(1-x)^2$ obtain $y'' - \frac{2}{(1-x)^2}y = 0$. Thus $p = 0$ and $q = \frac{-2}{(1-x)^2}$. Both of these functions are analytic: first because it is a polynomial and the second since it is a rational function so both have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular. Since 1 is a singularity of q, the radius of convergence of the solutions is 1 (which is the distance from center 0 to singularity 1). So, the interval of convergence of the solutions is $(-1, 1)$.

Alternatively, you can argue that p and q are analytic since they have convergent power series expansions at $x = 0$. The expansion of p is $0 + 0x + 0x^2 + \ldots$. The expansion of q can be obtained differentiating $-2\frac{1}{1-x} = -2(1 + x + x^2 + \ldots)$. This series is convergent on $(-1, 1)$ and so the expansion of q is convergent on $(-1, 1)$. Thus, the solutions are convergent on $(-1, 1)$, too.

Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives in the equation produces the recursive equation $a_{n+2} = \frac{2na_{n+1}-(n-2)a_n}{(n+2)}$ for $n = 0, 1, \ldots$ which produces two fundamental solutions $y_1 = 1 - 2x + x^2 = (1-x)^2$ and $y_2 = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Thus, the general solution is $y = c_1(1-x)^2 + c_2\frac{1}{1-x}$.

(b) For this equation $p = -2x$ and $q = 4$. Both of these are analytic since their derivatives are constant functions. Alternatively, you can deduce that p and q are analytic since their power series expansions are $p = 0 - 2x + 0x^2 + \ldots$ and $q = 4 + 0x + 0x^2 + \ldots$. So the point $x = 0$ is regular. Both expansions for p and q are convergent for every x and so the series solution is convergent for every x too.

Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives into the equation produces a recursive equation $a_{n+2} = \frac{(2n-4)a_n}{(n+1)(n+2)}$ for $n = 0, 1, \ldots$ Considering even-indexed coefficients obtain one fundamental solution $y_1 = 1 - 2x^2$. Considering odd-indexed coefficients obtain $y_2 = x - \frac{2}{3!}x^3 - \frac{2\cdot 2}{5!}x^5 - 2\frac{\cdot 2\cdot 4}{7!}x^7 \ldots$ The general solution is $y = c_1y_1 + c_2y_2$.