Math Methods 1
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Review for Exam 4

1. **Series Solutions. Regular-singular point**.

   (a) Consider the equation $4xy'' + 2y' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$ and determine the interval of convergence.

   (b) Consider the equation $xy'' - xy' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the closed form of one solution and list first few terms of the second solution.

   (c) Consider the equation $x^2y'' - xy' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$.

   (d) Consider the equation $3x^2y'' - 4xy' + 2y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$ and determine the interval of convergence.

   (e) Consider the equation $x^2y'' + 2xy' - x^2y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the series form of one solution and write down the form of the second solution (do not need to solve for coefficients of y_2).

   (f) Consider the equation $x(1-x)y'' + (1-x)y' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the closed form of one solution and write down the form of the second solution (do not need to solve for coefficients of the second solution). Determine the interval of convergence.

2. **Groups.**

   (a) Prove that the set of real numbers different from $-\frac{1}{3}$ is a group under the following operation
   $$a * b = a + b + 3ab.$$
   Determine whether it a group if $-\frac{1}{3}$ is included in the set.

   (b) Consider 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a \neq 0$ and $c \neq 0$. These matrices are called **upper triangular invertible** matrices. Show that the set of such matrices with matrix multiplication is a group.

   (c) Write down the Cayley tables for the following. (i) $C_2 \times C_3$, (ii) $C_2 \times C_2 \times C_2$, (iii) D_5.

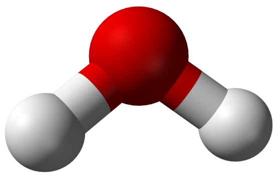
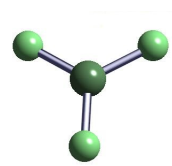
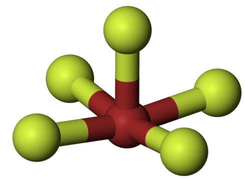
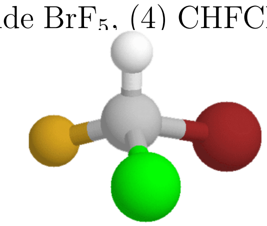
   (d) Produce all isomorphism classes of abelian groups of order 24. Do the same for groups of order 36.

   (e) Determine if the following pairs of groups are isomorphic. If they are, produce the isomorphism. If they are not, explain why.

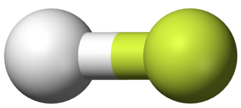
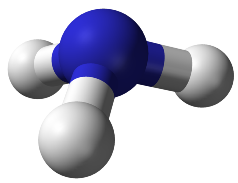
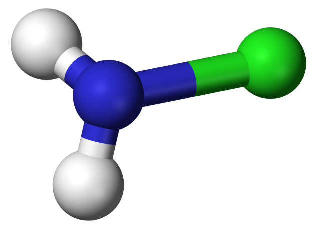
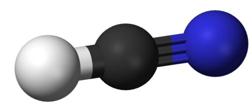
   (i) C_3 and D_3. (ii) C_6 and D_3.
   (iii) S_3 and D_3. (iv) S_n and D_n for $n > 3$.

(f) Describe the point groups of the following molecules. Write down the presentations of the point groups. Identify each group element as a symmetry operation.

(1) Water H_2O, (2) Boron trifluoride BF_3, (3) Bromine Pentafluoride BrF_5, (4) CHFClBr,

(5) Hydrogen chloride HCl, (6) Ammonia NH_3, (7) Chloramine NH_2Cl, (8) Hydrogen cyanide HCN.

Solutions.

1. Series solutions. More detailed solutions can be found on the class handouts.

(a) The point $x = 0$ is not regular since $p = \frac{1}{2x}$ and $q = \frac{1}{4x}$ are not defined at $x = 0$. However, $x = 0$ is a regular-singular point since the equation can be written as $x^2y'' + \frac{x}{2}y' + \frac{x}{4}y = 0$ so $\overline{p} = \frac{1}{2} = \frac{1}{2} + 0x + 0x^2 + \ldots$ and $\overline{q} = \frac{x}{4} = 0 + \frac{1}{4}x + 0x^2 + 0x^3 + \ldots$ are analytic at $x = 0$ and converge for any point x. So the interval of convergence is $(-\infty, \infty)$.

Search for solutions in the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Obtain the indical equation $r(2r-1) = 0 \Rightarrow r = 0$ or $r = \frac{1}{2}$. Hence, this is the first case since the difference $r_1 - r_2$ is not an integer. The case $r = 0$ produces one fundamental solution $y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n = \cos\sqrt{x}$. The case $r = \frac{1}{2}$ produces another fundamental solution $y = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^{1/2})^{2n+1} = \sin\sqrt{x}$. Thus, the general solution is $y = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x}$.

(b) $\overline{p} = -x$ and $\overline{q} = x$ are analytic (power series expansions are $p = 0 - 1x + 0x^2 + 0x^3 + \ldots$ and $q = 0 + 1x + 0x^2 + 0x^3 + \ldots$ and are convergent at any x). So $x = 0$ is a regular-singular point. Find the solutions in the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. The indical equation is $r(r-1) = 0 \Rightarrow r_1 = 1$ and $r_2 = 0$. So, the difference $r_1 - r_2$ is an integer. If $r = 1$, obtain that $a_{n+1} = \frac{n a_n}{(n+2)(n+1)}$ for $n = 1, 2 \ldots$ Thus $a_1 = a_2 = a_3 = \ldots = 0$, taking $a_0 = 1$ obtain $y_1 = x(1 + 0) = x$. The other solution can be found in the form $y = cx \ln x + \sum_{n=0}^{\infty} b_n x^n$. Plugging this and its derivatives in the equation reduces to $b_0 = -c$ and $b_n = \frac{c}{(n-1)\cdot n!}$ for $n = 2, 3, \ldots$ Taking $b_1 = 0$ and $c = 1$ produces $y_2 = x \ln x - 1 + \sum_{n=2}^{\infty} \frac{x^n}{(n-1)\cdot n!}$. The general solution is $y = c_1 x + c_2 y_2$.

(c) $\overline{p} = -1$ and $\overline{q} = 1$ are analytic (all derivatives are zero or, alternatively, constant functions have convergent power series expansion).

The indical equation is $(r-1)(r-1) = 0 \Rightarrow r_1 = r_2 = 1$ so the difference of the roots is zero. For $r = 1$, obtain $y_1 = x(1 + 0) = x$. The second solution has the form $y = x \ln x + \sum_{n=0}^{\infty} b_n x^{n+2}$. Calculate that $b_n = 0$ for any n. So $y_2 = x \ln x$ and the general solution is $y = c_1 x + c_2 x \ln x$.

(d) $\overline{p} = -\frac{4}{3}$ and $\overline{q} = \frac{2}{3}$. Constant functions have power series expansions which converges at every point. So $x = 0$ is a regular-singular point and the series solution is convergent on $(-\infty, \infty)$. The indical equation is $3r^2 - 7r + 2 = 0 \Rightarrow r_1 = 2$ and $r_2 = \frac{1}{3}$. So, the difference $r_1 - r_2$ is not an integer. For $r = 2$, obtain the polynomial solution $y_1 = x^2(1+0) = x^2$. For $r = \frac{1}{3}$, obtain the solution $y_2 = x^{1/3}(1+0) = x^{1/3}$. So, the general solution is $y = c_1x^2 + c_2x^{1/3}$.

(e) $\overline{p} = 2$ and $\overline{q} = -x^2$. These functions are analytic with convergent power series expansions are $p = 2 + 0x + 0x^2 + \ldots$ and $q = 0 + 0x - 1x^2 + 0x^3 + 0x^4 \ldots$. So $x = 0$ is a regular-singular point. The indical equation is $r(r-1) + 2r = 0 \Rightarrow r(r+1) = 0 \Rightarrow r_1 = 0$ and $r_2 = -1$ So, the difference $r_1 - r_2$ is an integer. For $r = 0$, $y_1 = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}$. The second solution has the form $y_2 = cx \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n$.

(f) Write the equation as $x^2y'' + xy' + \frac{x}{1-x}y = 0$. For this form, $\overline{p} = 1$ and $\overline{q} = \frac{x}{1-x}$. $x = 1$ is a singularity of $\overline{q}$ so the radius of convergence is 1 (=distance from center 0 to singularity 1) and, hence, the interval of convergence is $(-1, 1)$. Alternatively, $\overline{q} = x\frac{1}{1-x} = x(1 + x + x^2 + \ldots) = x + x^2 + x^3 + \ldots$ convergent on $(-1, 1)$. So, $x = 0$ is a regular-singular point and the solutions are convergent on interval $(-1, 1)$. The indical equation is $r^2 = 0 \Rightarrow r_1 = r_2 = 0$. So, this is the third case. For $r = 0$, obtain $y_1 = 1 - x$. The second solution has the form $y_2 = (1-x)\ln x + x\sum_{n=0}^{\infty} b_n x^n$ and the general solution is $y = c_1(1-x) + c_2y_2$.

2. Groups. More detailed solutions can be found on the class handouts.

(a) Check that the axioms A1–A4 hold. For A1, show that if a and b are real numbers different from $-\frac{1}{3}$, the product $a * b = a + b + 3ab$ is a real number different from $-\frac{1}{3}$ as well since

$$a * b = a + b + 3ab = -\tfrac{1}{3} \Rightarrow a + b + 3ab + \tfrac{1}{3} = 0 \Rightarrow a + \tfrac{1}{3} + b(1+3a) = 0 \Rightarrow$$
$$a + \tfrac{1}{3} + 3b(\tfrac{1}{3} + a) = 0 \Rightarrow (a + \tfrac{1}{3})(1+3b) = 0 \Rightarrow a + \tfrac{1}{3} = 0 \text{ or } 1 + 3b = 0 \Rightarrow a = -\tfrac{1}{3} \text{ or } b = -\tfrac{1}{3}.$$

For A2, show that both $(a*b)*c$ and $a*(b*c)$ are equal to $a+b+c+3ab+3ac+3bc+9abc$. For A3, show that $x = 0$ is a solution of $a*x = x*a = a$. Thus, the group identity element is 0. For A4, show that $x = \frac{-a}{1+3a}$ is a solution of $a * x = x * a = 0$. x is well defined since $a \neq \frac{-1}{3}$ and $x \neq \frac{-1}{3}$ since otherwise you would have $1 = 0$.
If all real numbers are considered instead of all numbers different from $\frac{-1}{3}$, the axiom A4 would fail because the equation $\frac{-1}{3} * x = 0$ has no solution (show why that is so).

(b) Show that A1–A4 hold. For A1, show that the product of two upper triangular invertible matrices is again an upper triangular and invertible. For A2, show that associativity holds. For A3, show that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the solution of $AX = XA = A$ for any $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.
For A4, show that $X = \begin{bmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix}$ is the solution of $AX = XA = I$ for $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

(c) $C_3 \times C_2 = \langle a, b | a^3 = b^2 = 1, ab = ba \rangle$, $C_2 \times C_2 \times C_2 = \langle a, b, c | a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$, and $D_5 = \langle a, b | a^5 = 1, b^2 = 1, a^4b = ba \rangle$. For Cayley's tables, see pages 11 and page 24 of the handout "Groups".

(d) $24 = 8 \cdot 3 = 2^3 \cdot 3$. There are 3 non-isomorphic abelian groups of order 24.

1.	$C_3 \times C_2 \times C_2 \times C_2$	$\cong C_6 \times C_2 \times C_2$	
2.	$C_3 \times C_4 \times C_2$	$\cong C_{12} \times C_2$	$\cong C_6 \times C_4$
3.	$C_3 \times C_8 \cong C_{24}$		

$36 = 4 \cdot 9 = 2^2 \cdot 3^2$. There are 4 non-isomorphic abelian groups of order 36.

1.	$C_3 \times C_3 \times C_2 \times C_2$	$\cong C_6 \times C_3 \times C_2$	$\cong C_6 \times C_6$
2.	$C_3 \times C_3 \times C_4$	$\cong C_3 \times C_{12}$	
3.	$C_9 \times C_2 \times C_2$	$\cong C_{18} \times C_2$	
4.	$C_9 \times C_4$	$\cong C_{36}$	

(e) (i) C_3 and D_3 are not isomorphic because one has 3 elements and the other has 6 elements.

(ii) C_6 and D_3 are not isomorphic because one is abelian and the other is not.

(iii) S_3 and D_3 are isomorphic. If 6 elements of S_3 are represented by maps f_1 to f_6 mapping (1, 2, 3) to (1, 2, 3), (2, 3, 1), (3, 1, 2), (1, 3, 2), (3, 2, 1), and (2, 1, 3), respectively, then f_1 is the identity, the order of f_2 and f_3 is 3 and the order of f_4, f_5 and f_6 is 2. If we denote $f_1 = 1$, $f_2 = a$ and $f_4 = b$, then $f_3 = f_2^2 = a^2$, $ab = f_2 f_4 = f_5$, $a^2 b = f_2^2 f_4 = f_6$, and $ba = f_4 f_2 = f_6 = a^2 b$. Thus, S_3 can be presented by $\langle a, b | a^3 = 1, b^2 = 1, ba = a^2 b\rangle$ which is the presentation of D_3 as well. So, the groups are isomorphic.

(iv) S_n and D_n are not isomorphic for $n > 3$ because one has $2n$ and the other $n!$ elements. $n!$ is larger than $2n$ for $n > 3$.

(f) (1) Water H_2O. This molecule has the following symmetries: identity E, rotation for 180 degrees a reflections with the respect to the vertical plane b and their product ab. Thus, this group is isomorphic to $C_2 \times C_2 = \langle a, b | a^2 = b^2 = 1, ba = ab\rangle$.

(2) Boron trifluoride BF_3. There are two nontrivial rotations: a rotation for 120 and a^2 rotation for 240 degrees and the symmetries with respect to vertical plane b and horizontal plane c. Since a and c commute and $ba = a^2 b$, we have $D_3 \times C_2 = \langle a, b, c | a^3 = 1, b^2 = 1, c^2 = 1, ca = ac, cb = bc, ba = a^2 b\rangle$.

(3) Bromine Pentafluoride BrF_5. Four fluor atoms line in the same plane forming the vertices of a square. Bromine atom is in the center of that square and the remaining fluor atom is directly above the bromine. Because of that fifth fluor, there are no symmetries with respect to horizontal plane and the symmetry group corresponds to the group of symmetries of a square is $D_4 = \langle a, b | a^4 = 1, b^2 = 1, ba = a^3 b\rangle$.

(4) CHFClBr has 5 different atoms so there is just the trivial symmetry. Thus, the point group is the trivial (one element) group $C_1 = \{1\}$.

(5) Hydrogen chloride HCl. All the rotations for any angle between 0 and 2π with 0 and 2π identified are the elements of the point group of this molecule. These rotations constitute the group denoted $C_\infty = SO(2, R)$. There is also the symmetry with respect to the vertical plane b. Since b does not commute with the rotations, the group is D_∞.

(6) Ammonia molecule has the same symmetries as the equilateral triangle. It has no symmetries with respect to the horizontal plane. Thus, the point group is $D_3 = \langle a, b | a^3 = 1, b^2 = 1, ba = a^2 b\rangle$.

(7) Chloramine NH_2Cl. The only non-identity group element is a single symmetry of order 2. So, the point group has two elements and so it is $C_2 = \langle a | a^2 = 1\rangle$.

(8) Hydrogen cyanide HCN is a linear molecule so all the rotations for any angle between 0 and 2π are in its point group. There is also symmetry b with respect to the vertical plane (vertical if the molecule stands "upright"). Since b does not commute with the rotations, the group is D_∞.