Math Methods 1
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Series Solutions of Ordinary Differential Equations

Let us consider a linear differential equation of the second order

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

where a, b, c, and g are some real-valued functions.

Recall that the equation $a(x)y'' + b(x)y' + c(x)y = 0$ is the **homogeneous part** of the above relation. If y_h is the solution of the homogeneous part of the non-homogeneous equation $a(x)y'' + b(x)y' + c(x)y = g(x)$, the particular solution y_p can be found using the variation of parameters method and the solution of the homogeneous part. In this case, the general solution of the equation has the form $y = y_h + y_p$.

Knowing the variation of parameters method, the problem boils down to finding the solution of the homogeneous part. If the functions $a(x), b(x)$ and $c(x)$ are constant, methods of Differential Equations course can be used. However, if these functions are not constant, another method is needed.

So, we shall concentrate on methods for finding the solution of the homogeneous equation $a(x)y''+ b(x)y' + c(x)y = 0$ if the functions $a(x), b(x)$ or $c(x)$ are not constant.

There are certain methods for finding this solution in some particular cases. For example,

- **Legendre's linear equation** $a(px + q)^2y'' + b(px + q)y' + cy = 0$ where a, b, c, p and q are constants. This equation can be reduced to an equation with constant coefficients using the substitution $px + q = e^t$. If $p = 1$ and $q = 0$, the equation $ax^2y'' + bxy' + cy = 0$ is called **Euler's equation**.

- **Exact equations**. If the left hand side of the equation $a(x)y'' + b(x)y' + c(x)y = 0$ is a derivative of another equation, it is said to be **exact**. It can be shown that an equation is exact if $c(x) - b'(x) + a''(x) = 0$. Note that if $a(x) = 0$, we can write the equation in the form $b(x)dy = -c(x)ydx$ and denote $Q = b$ and $P = -cy$, then the condition $c(x) - b'(x) = 0$ is equivalent to $P_y = Q_x$ which is the criterion of exactness for the first order differential equation.

- **Partially known complementary functions**. If one solution y_1 of the homogeneous equation $a(x)y''+b(x)y'+c(x)y = 0$ is known, to find the second solution y_2 and the general solution $y = c_1y_1 + c_2y_2$ as well, assume that it has the form $y_2 = vy_1$ where v is an unknown function. Substituting the derivatives of y_2 in the equation yields a separable differential equation in v'. Solving for v' first and then finding v produces the second complementary function y_2 of the general solution.

However, none of the above methods are general. The goal of this section is to present a general method that produces the solution of the equation $a(x)y'' + b(x)y' + c(x)y = 0$ in a form of a **power series**. Having a solution in the form of a power series should not be considered as a downside –

most commonly occurring functions such as polynomials, exponential and trigonometric functions have power series extension.

Let us start by assuming that $a(x) \neq 0$ (so that the equation is really of the second order) and by dividing by $a(x)$ to obtain the form

$$y'' + p(x)y' + q(x)y = 0$$

Although we shall denote the independent variable as x suggesting it is a real variable, our following discussion and the methods presented are valid for **functions of complex variables** as well.

Regular, Singular and Regular-Singular Points

Let us consider the equation $y'' + p(x)y' + q(x)y = 0$ and let us assume that the function $p(x)$ and $q(x)$ are **analytic** that is, that they have power series expansions

$$p(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n \text{ and } q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n$$

centered at a point $x = x_0$ which are convergent on some interval containing x_0. In this case, the point $x = x_0$ is said to be a **regular (or ordinary) point**.

If the functions $p(x)$ or $q(x)$ are not analytic at $x = x_0$, then the point $x = x_0$ is said to be a **singular point**. In this case when it is possible to write the equation in the form

$$(x - x_0)^2 y'' + \overline{p}(x)(x - x_0)y' + \overline{q}(x)y = 0$$

where the functions $\overline{p}(x)$ and $\overline{q}(x)$ are analytic at $x = x_0$, the point $x = x_0$ is said to be a **regular-singular point**. If it is not possible to represent the equation in this way, the point $x = x_0$ is said to be **irregular or essential singularity.**

The following examples showcase some equations that appear in applications in physics, and the classifications of their relevant singularities.

Equation	singular points	type
Legendre $(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$	± 1	regular-singular
Chebyshev $(1 - x^2)y'' - xy' + p^2 y = 0$	± 1	regular-singular
Bessel $x^2 y'' + xy' + (x^2 - a^2)y = 0$	0	regular-singular
Laguerre $xy'' + (1 - x)y' + ay = 0$	0	regular-singular
Hermite $y'' - 2xy' + 2ay = 0$	none	all pts regular
Simple harmonic oscillator $y'' + a^2 y = 0$	none	all pts regular

Solutions about an ordinary point

If $x = x_0$ is a regular point of $y'' + p(x)y' + q(x)y = 0$, then every solution of this equation can be represented as a power series centered at x_0.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Note that in this case

$$y' = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n \text{ and}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n$$

The solution y can be obtained following the steps below.

1. Substitute y, y' and y'' in the equation.
2. Write the left hand side of the equation as a single power series.
3. Equate the coefficients of the series you obtained to zero. This produces a **recursive equation** that computes the terms of the sequence a_n.
4. If possible, obtain the sequence a_n as a function of n. Then obtain the solution y as a power series. If you can express this series as an elementary function, then it is said that the solution is in **closed form**. The close form will not always be possible to obtain.

   When finding the closed form of the solutions, the power series expansions of the following elementary functions may be useful:

   $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

   $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

5. The **radius of convergence** of this power series is the distance from the center $x = x_0$ to the next nearest singularity of p and q.

To illustrate this last point, consider the equation $y'' - \frac{1}{x-1} y = 0$ about the regular point $x = 0$. For this equation $p = 0$ and $q = \frac{-1}{x-1} = \frac{1}{1-x}$. Both of these functions are analytic: first because it is a polynomial and the second since it is a rational function so both have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular. Alternatively, the functions p and q are analytic since they have convergent power series expansions at $x = 0$. The expansion of p is $0 + 0x + 0x^2 + \dots$. The expansion of q is $\frac{1}{1-x} = 1 + x + x^2 + \dots$.

This series is convergent on $(-1,1)$ and so the power series solution of this equation is convergent on $(-1,1)$ too. Note that the radius of convergence is 1 corresponds exactly the distance from the center $x=0$ to the first singularity $x=1$.

To illustrate the method, we will start by an example that can be solved using much simpler approach. Thus, we stress that this example is here just to illustrate how the method works, not to illustrate the usefulness of the method.

Example 1 – "Fake" example. Consider the equation $y''-2y'+y=0$. Show that $x=0$ is a regular point of this equation. Then find the series solution at $x=0$ and express your answers in the closed form.

Solution. In this case $p=-2$ and $q=1$. The functions p and q are analytic since the derivatives of these functions (the constant functions 0) are continuous. Alternatively, you can say that p and q are analytic since their power series expansions at $x=0$ are $p=2+0x+0x^2+\ldots$ and $q=1+0x+0x^2+\ldots$ which are convergent for every value of x. Thus, the point $x=0$ is regular.

The solution of the equation can be found in the form

$$y=\sum_{n=0}^{\infty}a_nx^n.$$

Find the derivatives to be

$$y'=\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \text{ and}$$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

and substitute them into the equation to get $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=0\Rightarrow$

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n]x^n=0$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n=0\Rightarrow a_{n+2}=\frac{2(n+1)a_{n+1}-a_n}{(n+1)(n+2)} \text{ for all } n=0,1,\ldots$$

The above formula represents the recursive equation that computes the coefficients of the sequence a_n. Note that this depends on two initial conditions a_0 and a_1. Now you are looking for convenient choices of a_0 and a_1 to obtain two solutions. In this case, the next term is $a_2=\frac{2a_1-a_0}{1\cdot 2}$ and with $a_0=a_1$ this simplifies to $a_2=\frac{a_0}{2}$. Continuing computing the terms, you obtain $a_3=\frac{2a_0-a_0}{3\cdot 2}=\frac{a_0}{3!}$, $a_4=\frac{2(3)a_3-a_2}{4\cdot 3}=\frac{a_0}{4\cdot 3\cdot 2}=\frac{a_0}{4!}\ldots a_n=\frac{a_0}{n!}$. To simplify further assume that $a_0=1$ and obtain the first solution as

$$y_1=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots=\sum_{n=0}^{\infty}\frac{1}{n!}x^n=e^x.$$

Since you are looking for linearly independent second solution, make a fundamentally different choice of a_0 and a_1. For example, take $a_0=0$. In this case $a_2=\frac{2a_1}{2}=a_1$, $a_3=\frac{2(2)a_1-a_1}{3\cdot 2}=\frac{a_1}{2}$,

$a_4 = \frac{2(3)a_3 - a_2}{4 \cdot 3} = \frac{3a_1 - a_1}{4 \cdot 3} = \frac{a_1}{2 \cdot 3} = \frac{a_1}{3!}$, $a_5 = \frac{8a_4 - a_3}{5 \cdot 4} = \frac{\frac{8a_1}{6} - \frac{a_1}{2}}{5 \cdot 4} = \frac{5a_1}{5!} = \frac{a_1}{4!} \ldots a_n = \frac{a_1}{(n-1)!}$ or $a_{n+1} = \frac{a_1}{n!}$
Assuming that $a_1 = 1$, you obtain the second solution as

$$y_1 = x + x^2 + \frac{1}{2!}x^3 + \frac{1}{3!}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}x^{n+1} = x \sum_{n=0}^{\infty} \frac{1}{n!}x^n = xe^x.$$

Note that the radius of the convergence is infinity. We can make this conclusion either by noting that the radius of convergence of both series representing two solutions is infinity or, easier, by noting that the equation has no singular points.

Thus, the general solution is $y = c_1 e^x + c_2 x e^x$.

As we pointed out, the equation $y'' - 2y' + y = 0$ has constant coefficients so it could be solved by considering the characteristic equation $r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0$ and finding the solution in the form $y = c_1 e^x + c_2 x e^x$. So, we turn to more relevant examples – examples of equations that do not have constant coefficients.

Example 2 – "Real" Example. Consider the equation $(1-x)^2 y'' - 2y = 0$. Show that $x = 0$ is a regular point of this equation. Then find the series solution at $x = 0$. Write your solution in the closed form and determine the radius of the convergence of the solution.

Solution. Dividing by $(1-x)^2$ we obtain the form $y'' - \frac{2}{(1-x)^2}y = 0$ so that $p = 0$ and $q = \frac{-2}{(1-x)^2}$. Both of these functions are analytic: first because it is a polynomial and the second since it is a rational function so both have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular. Since 1 is a singularity of q, the radius of convergence of the solutions is 1 (which is the distance from the center 0 to the singularity 1). So, the interval of convergence of the solutions is $(-1, 1)$.

Alternatively, you can argue that p and q are analytic since they have convergent power series expansions at $x = 0$. The expansion of p is $0 + 0x + 0x^2 + \ldots$. The expansion of q can be obtained differentiating $-2\frac{1}{1-x} = -2(1 + x + x^2 + \ldots)$. This series is convergent on $(-1, 1)$ and so the expansion of q is convergent on $(-1, 1)$. Thus, the solutions are convergent on $(-1, 1)$, too.

The solution can be found in the form $y = \sum_{n=0}^{\infty} a_n x^n$ and the derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Substitute the function and its derivatives into the equation.

$$(1 - 2x + x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty} (n+1)n a_{n+1}x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 2(n+1)n a_{n+1} + n(n-1)a_n - 2a_n]x^n = 0.$$

Combine the terms with a_n and factor out $n+1$ to get

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + (n^2-n-2)a_n]x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + (n+1)(n-2)a_n]x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty}(n+1)[(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n]x^n = 0$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n = 0 \ \Rightarrow a_{n+2} = \frac{2na_{n+1} - (n-2)a_n}{(n+2)} \text{ for all } n = 0, 1, \ldots$$

Note that $a_2 = \frac{2a_0}{2} = a_0, \quad a_3 = \frac{2a_0+a_1}{3}, \quad a_4 = \frac{4a_3}{4} = a_3, \quad a_5 = \frac{6a_3-a_3}{5} = a_3, \quad a_6 = \frac{8a_3-2a_3}{6} = a_3\ldots$ Thus, choosing $2a_0 + a_1 = 0$ would produce $0 = a_3 = a_4 = a_5 = \ldots$. So the only non-zero coefficients are a_0, $a_1 = -2a_0$ and $a_2 = a_0$. This yields a **polynomial solution**. Choosing $a_0 = 1$ we obtain the first solution

$$y_1 = 1 - 2x + x^2 = (1-x)^2.$$

Choosing $a_1 = a_0$ we obtain $a_2 = a_0, a_3 = \frac{2a_0+a_0}{3} = a_0, a_4 = a_3 = a_0, \ldots a_n = a_0$. Choose $a_0 = 1$ for simplicity again and obtain the second solution

$$y_2 = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Note that the radius of convergence of this series is 1 so it converges for $-1 < x < 1$ just as we projected at the beginning.

The two solutions y_1 and y_2 are obviously independent (since one is a polynomial and the other is not) so the general solution is

$$y = c_1(1-x)^2 + c_2\frac{1}{1-x}.$$

Practice Problems.

1. Consider the differential equation $y'' - y = 0$. Show that $x = 0$ is a regular point of this equation. Find the series solutions of the given equation about $x = 0$. Express your answers in closed form.

2. Consider Hermite equation $y'' - 2xy' + 4y = 0$. Show that $x = 0$ is a regular point of this equation. Find the series solutions of the given equation about $x = 0$. Find the closed form of one solution and list first few terms of the second solution. Determine the interval of convergence.

Solutions.

1. For this equation $p = 0$ and $q = -1$ so both of these functions are analytic (all derivatives are constant or, alternatively, the convergent power series expansions are $p = 0 + 0x + 0x^2 + \ldots$ and $q = -1 + 0x + 0x^2 + \ldots$). Thus, $x = 0$ is a regular point and the solution can be found in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Find the derivatives to be

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

and substitute them into the equation.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1) a_{n+2} - a_n = 0 \text{ for all } n = 0, 1, \ldots \Rightarrow a_{n+2} = \frac{a_n}{(n+1)(n+2)} \text{ for all } n = 0, 1, \ldots$$

The above formula represents the recursive equation that computes the coefficients of the sequence a_n. In order to find the solution of the recursive equation, consider the first couple of terms.

$$a_2 = \frac{a_0}{1 \cdot 2}, \quad a_4 = \frac{a_2}{3 \cdot 4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4} \quad \Rightarrow \quad a_{2n} = \frac{a_0}{(2n)!}$$

and

$$a_3 = \frac{a_1}{2 \cdot 3}, \quad a_5 = \frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5} \quad \Rightarrow \quad a_{2n+1} = \frac{a_1}{(2n+1)!}$$

Recall that we are looking for two linearly independent solutions y_1 and y_2 to be produced by these formulas. For example,
if we take $a_0 = a_1 = 1$, we obtain $a_n = \frac{1}{n!} \quad \Rightarrow \quad y_1 = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ and
If we take $a_0 = -a_1 = 1$, we obtain $a_n = \frac{(-1)^n}{n!} \quad \Rightarrow \quad y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}$.
Alternatively, by taking $a_0 = 1$ and $a_1 = 0$, you obtain $\cosh(x)$ and by taking $a_0 = 0$ and $a_1 = 1$, you obtain $\sinh(x)$.

Note that the radius of the convergence is infinity because the equation has no singular points.

Thus, the general solution is $y = c_1 e^x + c_2 e^{-x}$ (alternatively $y = c_1 \sinh(x) + c_2 \cosh(x)$).

2. Here $p = -2x$ and $q = 4$ are analytic since their derivatives are constant functions. Alternatively, you can deduce that p and q are analytic since their power series expansions are $p = 0 - 2x + 0x^2 + \ldots$ and $q = 4 + 0x + 0x^2 + \ldots$. So the point $x = 0$ is regular. Both expansions for p and q are convergent for every x and so the series solution is convergent for every x too.

   The solution can be found in the form

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substitute these into the equation and get

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty} na_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0 \;\Rightarrow$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2na_n + 4a_n\right]x^n = 0.$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1)a_{n+2} - 2na_n + 4a_n \text{ for all } n = 0, 1, \ldots \Rightarrow a_{n+2} = \frac{(2n-4)a_n}{(n+1)(n+2)} \text{ for all } n = 0, 1, \ldots$$

Thus the even-indexed coefficients depends on a_0 and the odd-indexed coefficients on a_1. We can obtain two linearly independent solutions by taking $a_0 = 1$ and $a_1 = 0$ for y_1 and $a_0 = 0$ and $a_1 = 1$ for y_2.

In the first case, note that $a_2 = \frac{-4}{2} = -2$ and $a_4 = 0$. Thus $a_6 = a_8 = a_{10} = \ldots = 0$. So y_1 is a polynomial $y_1 = 1 - 2x^2$.

In the second case, $a_3 = \frac{-2}{2\cdot 3}$, $a_5 = \frac{-2\cdot 2}{2\cdot 3\cdot 4\cdot 5} = \frac{-2\cdot 2}{5!}$, $a_7 = \frac{-2\cdot 2\cdot 4}{7!} \ldots$ Thus $y_2 = x - \frac{2}{3!}x^3 - \frac{2\cdot 2}{5!}x^5 - 2\frac{\cdot 2\cdot 4}{7!}x^7 \ldots$. The general solution $y = c_1y_1 + c_2y_2$ is convergent on $(-\infty, \infty)$.

Solutions about a regular-singular point

Recall that $x = x_0$ is a **regular-singular point** of the equation

$$(x - x_0)^2 y'' + \overline{p}(x)(x - x_0)y' + \overline{q}(x)y = 0$$

if the functions $\overline{p}(x)$ and $\overline{q}(x)$ are analytic at an interval containing $x = x_0$.

For simplicity, we can assume that $x_0 = 0$. If that is not the case, the substitution $t = x - x_0$ can convert the equation with a regular-singular point $x = x_0$ to an equation with regular-singular point $t = 0$.

With the assumption that $x_0 = 0$, we search for a solution in the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

for some number r. This series is called **Frobenius series**. We can assume that $a_0 \neq 0$ since otherwise we can redefine a_1 or some higher coefficient as a_0.

For $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, we have that

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \text{ and } y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

The solution y can be obtained following the steps below.

1. Substitute y, y' and y'' in the equation.

2. Write the left hand side of the equation as a single power series.

3. Consider the coefficient with *the lowest power of* x (equivalently, cancel the smallest power of x and then plug $x = 0$ to obtain this coefficient). Equate this coefficient to zero. The quadratic equation in r that you obtained in this way is called an **indical equation**. Let r_1 and r_2 denote the two solutions of the indical equation, called the **indices**, and let $r_1 \geq r_2$. We distinguish three relevant cases.

   (a) The difference $r_1 - r_2$ is not an integer. In this case, the two linearly independent solutions y_1 and y_2 are given by

   $$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} b_n x^n.$$

   (b) The difference $r_1 - r_2$ is a nonzero integer. If r_1 is the larger root, the two linearly independent solutions y_1 and y_2 are given by

   $$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = c y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n.$$

(c) The difference $r_1 - r_2$ is zero. In this case, the two linearly independent solutions y_1 and y_2 are given by

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^{r_1+1}\sum_{n=0}^{\infty} b_n x^n.$$

The coefficients a_n and b_n can be determined from two recursive equations obtained in the same way as in the case of a regular point – by substituting the solution $y = x^r \sum_{n=0}^{\infty} a_n x^n$ in the equation for two values r_1 and r_2.

4. Just in the case of a regular point, if you can express these series as elementary functions, then the solution is said to be in the **closed form**. The close form will not always be possible to obtain. Also, the **radius of convergence** of this power series is the distance from the center $x = x_0$ to the next nearest singularity of $\overline{p}$ and $\overline{q}$.

In the second or third case, the form of the second solution listed above can be obtained using a method known as **the derivative method**. One of your project topics asks you to look into this method in more details.

The following three examples illustrate these three cases.

Example 1 – Case 1 Example. Consider the equation $4xy'' + 2y' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$ and determine the interval of convergence.

Solution. The point $x = 0$ is not regular since $p = \frac{1}{2x}$ and $q = \frac{1}{4x}$ are not defined at $x = 0$. However, $x = 0$ is a regular-singular point since the equation can be written as $x^2y'' + \frac{x}{2}y' + \frac{x}{4}y = 0$ so $\overline{p} = \frac{1}{2} = \frac{1}{2} + 0x + 0x^2 + \ldots$ and $\overline{q} = \frac{x}{4} = 0 + \frac{1}{4}x + 0x^2 + 0x^3 + \ldots$ are analytic at $x = 0$ and converge for any point x. Thus,

$$y = x^r\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \text{ and } y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}.$$

Substitute into the equation to get

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

The smallest power of x appears in the first two sums for $n = 0$ and it is x^{r-1}. The coefficient with x^{r-1} is $4r(r-1)a_0 + 2ra_0$. This gives you the indical equation

$$4r(r-1)a_0 + 2ra_0 = 0 \Rightarrow 2a_0(2r^2 - 2r + r) = 0 \Rightarrow 2r^2 - r = 0 \Rightarrow r(2r-1) = 0 \Rightarrow r = 0 \text{ or } r = \frac{1}{2}.$$

Thus, this is the first case since the difference $r_1 - r_2$ is not an integer.

Let us first consider the case $r = 0$ which will yield the first solution $y = x^0 \sum_{n=0}^{\infty} a_n x^n$ from the equation

$$\sum_{n=0}^{\infty} 4n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} n[4(n-1)+2]a_n x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0 \Rightarrow n(4(n-1)+2)a_n + a_{n-1} = 0 \Rightarrow n(4n-2)a_n = -a_{n-1}$$

This produces the recursive relation $a_n = \frac{-a_{n-1}}{n(4n-2)} = \frac{-a_{n-1}}{2n(2n-1)}$ for $n = 1, 2, \ldots$ Choosing $a_0 = 1$ produces $a_1 = \frac{-1}{2}$, $a_2 = \frac{-a_1}{4\cdot 3} = \frac{1}{4\cdot 3\cdot 2} = \frac{1}{4!}$, $a_3 = \frac{-1}{6!} \ldots a_n = \frac{(-1)^n}{(2n)!}$. Thus, the first solution is

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$$

Comparing this sum with the series expansion for $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ we conclude that $y_1 = \cos\sqrt{x}$.

Let us consider now the case $r = \frac{1}{2}$ which yields the second solution $y = x^{1/2} \sum_{n=0}^{\infty} a_n x^n$ from the equation

$$\sum_{n=0}^{\infty} 4(n+\frac{1}{2})(n+\frac{1}{2}-1)a_n x^n + \sum_{n=0}^{\infty} 2(n+\frac{1}{2})a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} [(2n+1)(2n-1)+(2n+1)]a_n x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0 \Rightarrow [(2n+1)(2n-1)+(2n+1)]a_n + a_{n-1} = 0 \Rightarrow$$

$$(2n+1)2na_n = -a_{n-1} \Rightarrow a_n = \frac{-a_{n-1}}{2n(2n+1)} \text{ for } n = 1, 2, \ldots$$

Choosing $a_0 = 1$ produces $a_1 = \frac{-1}{2\cdot 3} = \frac{-1}{3!}$, $a_2 = \frac{-a_1}{5\cdot 4} = \frac{1}{5\cdot 4\cdot 3\cdot 2} = \frac{1}{5!}$, $a_3 = \frac{-1}{7!} \ldots a_n = \frac{(-1)^n}{(2n+1)!}$. Thus, the second solution is

$$y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n$$

Comparing this sum with the series expansion for $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ we conclude that $y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^{1/2})^{2n+1} = \sin\sqrt{x}$.

Thus, the general solution is $y = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x}$. The functions $\overline{p}$ and $\overline{q}$ converge on $(-\infty, \infty)$ and so the interval of convergence of the solutions is also $(-\infty, \infty)$.

Example 2 – Case 2 Example. Consider the equation $x^2y'' + xy' - y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$.

Solution. Note that $x = 0$ is regular-singular point since $\overline{p} = 1$ and $\overline{q} = -1$ are analytic (all derivatives are zero or, alternatively, constant functions have convergent power series expansion).

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \text{ and } y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}.$$

Substitute into the equation to get $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \Rightarrow \sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-1]a_n x^{n+r} = 0$

Equating the first term with zero produces the indical equation

$$r(r-1)+r-1=0 \Rightarrow r^2-1=0 \Rightarrow (r-1)(r+1)=0 \Rightarrow r_1=1, r_2=-1$$

so we are dealing with the case when the difference of the roots is a non-negative integer.

Plugging that $r=1$ in the equation gives us

$$x\sum_{n=0}^{\infty}[(n+1)n+(n+1)-1]a_nx^n=0 \Rightarrow [n^2+2n]a_n=0 \Rightarrow n(n+2)a_n=0, n=0,1,2,\ldots$$

Since the expression $n(n+2)$ is not zero for any positive value of n, we conclude that $a_n=0$ for all $n=1,2,\ldots$. Thus, all the coefficients a_n are zero except possibly the first one, a_0. Taking $a_0=1$ you obtain the first solution $y_1=x(1+0)=x$.

The second solution has the form $y=cy_1\ln x+x^{r_2}\sum_{n=0}^{\infty}b_nx^n=cx\ln x+x^{-1}\sum_{n=0}^{\infty}b_nx^n=cx\ln x+\sum_{n=0}^{\infty}b_nx^{n-1}$. The derivatives are

$y'=c+c\ln x+\sum_{n=0}^{\infty}(n-1)b_nx^{n-2}$ and $y''=\frac{c}{x}+\sum_{n=0}^{\infty}(n-1)(n-2)b_nx^{n-3}$.

Plug the function and the derivatives into the equation and obtain

$$cx+\sum_{n=0}^{\infty}(n-1)(n-2)b_nx^{n-1}+cx+cx\ln x+\sum_{n=0}^{\infty}(n-1)b_nx^{n-1}-cx\ln x-\sum_{n=0}^{\infty}b_nx^{n-1}=0$$

The terms with $\ln x$ cancel and the remaining terms in $2cx+\sum_{n=0}^{\infty}[(n-1)(n-2)+(n-1)-1]b_nx^{n-1}$ have to be equal to zero. Simplify to get $2cx+\sum_{n=0}^{\infty}(n^2-3n+2+n-2)b_nx^{n-1}=0 \Rightarrow$

$$2cx+\sum_{n=0}^{\infty}n(n-2)b_nx^{n-1}=0 \Rightarrow 2cx-b_1+3b_3x^2+4(2)b_4x^3+5(3)b_5x^4+\ldots=0$$

Considering the coefficients with each term we obtain that $c=0$, $b_1=0$ and $b_3=b_4=b_5\ldots=0$. Thus, b_0 and b_2 are the only two possible nonzero coefficients. Thus, $y_2=\frac{1}{x}(b_0+b_2x^2)=\frac{b_0}{x}+b_2x$. Since the last term is a constant multiple of the first solution, we can take $b_2=0$. Taking $b_0=1$ for simplicity, we obtain the second solution $y_2=\frac{1}{x}$.

Thus, the general solution is $y=c_1x+c_2\frac{1}{x}$.

Example 3 – Case 3 Example. Consider the equation $x^2y''-xy'+y=0$. Show that $x=0$ is a regular-singular point. Then find the closed form of the series solutions about $x=0$.

Solution. Note that $x=0$ is regular-singular point since $\overline{p}=-1$ and $\overline{q}=1$ are analytic (all derivatives are zero or, alternatively, constant functions have convergent power series expansion).

$$y=x^r\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+r} \Rightarrow y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} \text{ and } y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}.$$

Substitute into the equation to get $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r}=0 \Rightarrow$

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)-(n+r)+1]a_nx^{n+r}=0$$

Equate the coefficient of the smallest power of x with zero to produce the indical equation

$$r(r-1)-r+1=0 \Rightarrow (r^2-2r+1)=0 \Rightarrow (r-1)(r-1)=0 \Rightarrow r_1=r_2=1$$

so we are dealing with the case when the difference of roots is zero.

Plugging that $r=1$ in the equation gives us

$$x\sum_{n=0}^{\infty}[(n+1)n-(n+1)+1]a_nx^n=0 \Rightarrow [n^2+n-n-1+1]a_n=0 \Rightarrow n^2a_n=0$$

Thus, all the coefficients a_n are zero except the first one, a_0 when $n=0$. Take $a_0=1$ and obtain the first solution $y_1=x(1+0)=x$.

The second solution has the form $y = y_1\ln x + x^{r_1+1}\sum_{n=0}^{\infty} b_nx^n = x\ln x + x^2\sum_{n=0}^{\infty} b_nx^n = x\ln x+\sum_{n=0}^{\infty} b_nx^{n+2}$. The derivatives are

$y' = 1+\ln x+\sum_{n=0}^{\infty}(n+2)b_nx^{n+1}$ and $y''=\frac{1}{x}+\sum_{n=0}^{\infty}(n+2)(n+1)b_nx^n$.

Plug the function and the derivatives into the equation and obtain

$$x+\sum_{n=0}^{\infty}(n+2)(n+1)b_nx^{n+2}-x-x\ln x-\sum_{n=0}^{\infty}(n+2)b_nx^{n+2}+x\ln x+\sum_{n=0}^{\infty}b_nx^{n+2}=0$$

All the "non-series" terms cancel and the remaining series $\sum_{n=0}^{\infty}[(n+2)(n+1)b_n-(n+2)b_n+b_n]x^{n+2}$ has to have zero terms. Thus $(n+2)(n+1)b_n-(n+2)b_n+b_n=0 \Rightarrow$

$$(n^2+3n+2-n-2+1)b_n=0 \Rightarrow (n^2+2n+1)b_n=0 \Rightarrow (n+1)^2b_n=0 \Rightarrow b_n=0$$

for all n. Thus, the second solution is $y_2=x\ln x$ and the general solution is $y=c_1x+c_2x\ln x$.

The next example illustrate that in some special instances of the second case, the second solution can be obtained from the first one. In other words, both solutions can be obtained in the form $y=x^r\sum_{n=0}^{\infty}a_nx^n$.

Example 4. – "Lucky instance of the second case" Example. Consider the equation $x^2y''-2xy'+2y=0$. Show that $x=0$ is a regular-singular point. Then find the closed form of the series solutions about $x=0$.

Solutions. For this equation, $\overline{p}=-2$ and $\overline{q}=2$ and these two functions are analytic (all derivatives are zero, alternatively, constant functions have convergent power series expansion). So $x=0$ is a regular-singular point. Plugging the solution $y=\sum_{n=0}^{\infty}a_nx^{n+r}$ and its derivatives yield the equation

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)-2(n+r)+2]a_nx^{n+r}=0$$

Equating the first term with zero produces the indical equation $r(r-1)-2r+2=0 \Rightarrow r^2-3r+2=0 \Rightarrow (r-1)(r-2)=0$. So, the difference r_1-r_2 is an integer. The first solution can be obtained by taking the larger of the two r-values, $r=2$. In this case $[(n+2)(n+1)-2(n+2)+2]a_n=0 \Rightarrow (n^2+3n+2-2n-4+2)a_n=0 \Rightarrow (n^2+n)a_n=0 \Rightarrow n(n+1)a_n=0$. In this case all coefficients a_n are zero except possibly a_0. By taking $a_0=1$, we obtain the solution $y_1=x^2(1+0)=x^2$.

Consider now $r=1$. In this case $(n+1)na_n-2(n+1)a_n+2a_n=0 \Rightarrow (n^2+n-2n-2+2)a_n=0 \Rightarrow (n^2-n)a_n=0 \Rightarrow n(n-1)a_n=0$. In this case all coefficients a_n are zero except possibly a_0 (when

$n = 0$) and a_1 (when $n - 1 = 0$). Thus, $y_2 = x(a_0 + a_1 x) = a_0 x + a_1 x^2$. Since the second part has the form of the first solution, you can take $a_1 = 0$. Taking $a_0 = 1$ for simplicity, you obtain the solution $y_2 = x$ that is linearly independent from $y_1 = x^2$.

Thus, the general solution is $y = c_1 x^2 + c_2 x$.

Practice Problems.

1. Consider the equation $3x^2y'' - 4xy' + 2y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$ and determine the interval of convergence.

2. Consider the equation $xy'' - xy' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the closed form of one solution and list first few terms of the second solution.

3. Consider the equation $x^2y'' + 2xy' - x^2y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the series form of one solution and write down the form of the second solution (do not need to solve for coefficients of the second solution).

4. Consider the equation $x(1 - x)y'' + (1 - x)y' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the closed form of one solution and write down the form of the second solution (do not need to solve for coefficients of the second solution). Determine the interval of convergence.

Solutions.

1. For this equation, $\overline{p} = -\frac{4}{3}$ and $\overline{q} = \frac{2}{3}$. Constant functions have power series expansions which converges at every point. So $x = 0$ is a regular-singular point and the series solution is convergent on $(-\infty, \infty)$.

   Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives yield the equation

   $$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} 4(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

   Equating the coefficient of the smallest power of x, x^r in this case, with zero produces the indical equation $3r(r-1) - 4r + 2 = 0 \Rightarrow 3r^2 - 7r + 2 = 0 \Rightarrow r_1 = 2$ and $r_2 = \frac{1}{3}$. So, the difference $r_1 - r_2$ is not an integer.

   When $r = 2$ the equation becomes

   $$\sum_{n=0}^{\infty}[3(n+2)(n+1)a_n - 4(n+2)a_n + 2a_n]x^{n+2} = 0 \Rightarrow [3(n+2)(n+1) - 4(n+2) + 2]a_n = 0 \Rightarrow$$

   $(3n+5)na_n = 0 \Rightarrow a_n = 0$ for all $n > 0$. By taking $a_0 = 1$ we obtain the solution $y_1 = x^2(1+0) = x^2$.

   When $r = \frac{1}{3}$ the equation becomes

   $$\sum_{n=0}^{\infty}[3(n+\frac{1}{3})(n-\frac{2}{3})a_n - 4(n+\frac{1}{3})a_n + 2a_n]x^{n+2} = 0 \Rightarrow [3(n+\frac{1}{3})(n-\frac{2}{3}) - 4(n+\frac{1}{3}) + 2]a_n = 0 \Rightarrow$$

$(3n-5)na_n = 0 \Rightarrow a_n = 0$ for all $n > 0$. By taking $a_0 = 1$ we obtain the solution $y_2 = x^{1/3}(1+0) = x^{1/3}$.

So, the general solution is $y = c_1x^2 + c_2x^{1/3}$.

2. For this equation, $\overline{p} = -x$ and $\overline{q} = x$ are analytic (power series expansions are $\overline{p} = 0 - 1x + 0x^2 + 0x^3 + \ldots$ and $\overline{q} = 0 + 1x + 0x^2 + 0x^3 + \ldots$ and are convergent for any x). So $x = 0$ is a regular-singular point. Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives yield the equation

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_nx^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r} = 0$$

The smallest power of x is x^{r-1}. Its coefficient produces the indical equation $r(r-1) = 0 \Rightarrow r_1 = 1$ and $r_2 = 0$. So, the difference $r_1 - r_2$ is an integer.

One solution can be obtained by considering $r = 1$. In this case

$$\sum_{n=0}^{\infty} n(n+1)a_nx^n - \sum_{n=0}^{\infty}(n+1)a_nx^{n+1} + \sum_{n=0}^{\infty}a_nx^{n+1} = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+1} - (n+1)a_n + a_n]x^{n+1} = 0 \Rightarrow$$

$2a_1 = 0$ and $a_{n+1} = \frac{na_n}{(n+2)(n+1)}$ for $n = 1, 2\ldots$ Thus $a_1 = 0$ so $a_2 = a_3 = \ldots = 0$. Taking $a_0 = 1$ we obtain the first solution $y_1 = x(1+0) = x$.

The other solution can be found in the form $y = cx\ln x + \sum_{n=0}^{\infty} b_nx^n$. Plugging this and its derivatives in the equation produces

$$c + \sum_{n=0}^{\infty} n(n-1)b_nx^{n-1} - cx\ln x - cx - \sum_{n=0}^{\infty} nb_nx^n + cx\ln x + \sum_{n=0}^{\infty} b_nx^n = 0 \Rightarrow$$

$$c - cx + \sum_{n=0}^{\infty}[(n+1)nb_{n+1} - nb_n + b_n]x^n = 0 \Rightarrow$$

Equate the left-hand side terms with x^n for any n with zero.

For $n = 0$, we have that $c + b_0 = 0$. So, $b_0 = -c$.

For $n = 1$, we have that $-c + 2b_2 = 0$. So, $b_2 = \frac{c}{2}$

For $n = 2, 3, \ldots$, we have that $[(n+1)nb_{n+1} - nb_n + b_n \Rightarrow b_{n+1} = \frac{(n-1)b_n}{n(n+1)}$.

Thus $b_3 = \frac{b_2}{3\cdot 2} = \frac{c}{3\cdot 2\cdot 2} = \frac{c}{2\cdot 3!}$, $b_4 = \frac{2b_3}{4\cdot 3} = \frac{c}{4\cdot 3\cdot 3\cdot 2} = \frac{c}{3\cdot 4!}$, $b_5 = \frac{3b_4}{5\cdot 4} = \frac{c}{5\cdot 4\cdot 4\cdot 3\cdot 2} = \frac{c}{4\cdot 5!}\ldots$ and so $b_n = \frac{c}{(n-1)n!}$ for $n = 2, 3, \ldots$. We can take $b_1 = 0$ and $c = 1$ for simplicity. Thus $y_2 = x\ln x - 1 + \sum_{n=2}^{\infty} \frac{x^n}{(n-1)\cdot n!}$ and the general solution is $y = c_1x + c_2y_2$.

3. For this equation, $\overline{p} = 2$ and $\overline{q} = -x^2$. These functions are analytic with convergent power series expansions are $\overline{p} = 2 + 0x + 0x^2 + \ldots$ and $\overline{q} = 0 + 0x - 1x^2 + 0x^3 + 0x^4 \ldots$. So $x = 0$ is

a regular-singular point. Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives yield the equation

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

Equating the coefficient of the smallest power of x (the first term in the first two sums in this case) with zero produces the indical equation $r(r-1)+2r = 0 \Rightarrow r(r+1) = 0 \Rightarrow r_1 = 0$ and $r_2 = -1$ So, the difference $r_1 - r_2$ is an integer.

One solution can be obtained by considering $r = 0$. In this case
$\sum_{n=0}^{\infty} n(n-1)a_n x^n + 2\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \Rightarrow$
$2a_1 x + \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+2)a_{n+2} - a_n]x^{n+2} = 0 \Rightarrow$
$2a_1 = 0$ and $a_{n+2} = \frac{a_n}{(n+2)(n+3)}$ for $n = 0, 1, \ldots$ Thus $a_1 = 0$ and so $a_3 = a_5 = \ldots = 0$. The even terms are $a_2 = \frac{a_0}{3 \cdot 2} = \frac{a_0}{3!}, a_4 = \frac{a_0}{5!}, \ldots\ a_{2n} = \frac{a_0}{(2n+1)!}$. Taking $a_0 = 1$ we obtain the first solution $y_1 = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}$. The second solution has the form $y_2 = cx \ln x + x^{-1}\sum_{n=0}^{\infty} b_n x^n$. The general solution is $y = c_1 y_1 + c_2 y_2$.

4. Multiplying the equation by x and dividing by $1-x$ we obtain the form $x^2 y'' + xy' + \frac{x}{1-x} y = 0$. For this form, we can see that $\overline{p} = 1$ and $\overline{q} = \frac{x}{1-x}$. The function $\overline{p}$ is analytic with the power series $1 + 0x + 0x^2 + \ldots$ convergent for any n. The function $\overline{q} = x\frac{1}{1-x} = x(1 + x + x^2 + \ldots) = x + x^2 + x^3 + \ldots$ is convergent on interval $(-1, 1)$ since the expansion $1 + x + x^2 + \ldots$ of $\frac{1}{1-x}$ is convergent on $(-1, 1)$. So, $x = 0$ is a regular-singular point and the solutions are convergent on interval $(-1, 1)$. You can reach the same conclusion regarding the interval of convergence by noting that $x = 1$ is a singularity of $\overline{q}$ so the radius of convergence is 1 (which is the distance from the center 0 to the singularity 1) and, hence, the interval of convergence is $(-1, 1)$.

   Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives into the equation $xy'' - x^2 y'' + y' - xy' + y = 0$ produces

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} -$$

$$\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

   Equating the coefficient of the smallest power of x, x^{r-1} in this case, with zero produces the indical equation $r(r-1) + r = 0 \Rightarrow r^2 = 0 \Rightarrow r_1 = r_2 = 0$. So, this is the third case.

   For $r = 0$, the equation becomes $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty}[(n+1)n a_{n+1} - n(n-1)a_n + (n+1)a_{n+1} - n a_n + a_n]x^n = 0 \Rightarrow a_{n+1} = \frac{(n^2-1)a_n}{(n+1)^2} = \frac{(n-1)a_n}{n+1} \Rightarrow a_1 = -a_0,\ a_2 = 0 \Rightarrow a_3 = a_4 = a_5 = \ldots 0$. With $a_0 = 1$, we obtain $y_1 = 1 - x$.

   The second solution has the form $y_2 = (1-x)\ln x + x\sum_{n=0}^{\infty} b_n x^n$ and the general solution is $y = c_1(1-x) + c_2 y_2$.